Higher order symmetries: gauge covariant approach and gravitational anomalies

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Outline

1. Killing tensors
2. Classical conserved quantities
3. Quantum anomalies
4. Gauge covariant approach
5. Outlook

Killing tensors (1)

Killing vector

$$X_{(\mu;\lambda)} = 0\,.$$

Conformal Killing vector

$$Z_{(\lambda;\mu)} = \frac{1}{n} g_{\lambda\mu} Z^{\sigma}_{\ ;\sigma}\,.$$

Killing tensors (2)

Stackel-Killing tensor of order m

$$K_{\mu_1\cdots\mu_m} = K_{(\mu_1\cdots\mu_m)}\,,$$

$$K_{(\mu_1\cdots\mu_m;\sigma)} = 0\,.$$

Conformal Stackel-Killing tensor of order 2
(Tachibana 1969)

$$A_{(\mu\nu;\sigma)} = g_{(\mu\nu}A_{\sigma)}$$

with

$$A_\sigma = \frac{1}{n+2}(2A^\mu{}_{\sigma;\mu} + A^\mu{}_{\mu;\sigma})\,.$$

Killing tensors (3)

Killing-Yano tensor of order $p \leq n$ (Yano 1952)

$$\omega_{\mu_1 \cdots \mu_p} = \omega_{[\mu_1 \cdots \mu_p]} \,,$$

$$\omega_{\mu_1 \cdots \mu_{p-1}(\mu_p;\nu)} = 0.$$

Conformal Killing-Yano tensor of order 2

$$f_{\lambda\mu;\nu} + f_{\nu\mu;\lambda} = \frac{2}{n-1} \left(g_{\nu\lambda} f^{\sigma}_{\ \mu;\sigma} + g_{\mu(\lambda} f_{\nu)\ \ ;\sigma}^{\ \ \sigma} \right) \ .$$

Notation:

$$f_\kappa := f^{\sigma}_{\ \kappa;\sigma} \quad , \quad f^{\sigma}_{\ ;\sigma} = 0,$$

and consequently

$$f_{(\mu;\nu)} = \frac{n-1}{n-2} R_{\sigma(\mu} f_{\nu)}^{\ \ \sigma} \ .$$

Classical conserved quantities (1)

In classical mechanics a general function Q of position x and of momentum π is a constant of motion if and only if the Poisson bracket with the Hamiltonian

$$H = g_{\mu\nu}\pi^\mu\pi^\nu ,$$

vanishes

$$[H, Q]_P = \frac{\partial H}{\partial x^\mu}\frac{\partial Q}{\partial \pi_\mu} - \frac{\partial Q}{\partial x^\mu}\frac{\partial H}{\partial \pi_\mu} = 0 .$$

Suppose that γ is a geodesic with tangent vector u^μ which is interpreted as the momentum of a free particle of unit mass moving along geodesic.

Classical conserved quantities (2)

For a Killing vector or a Stackel-Killing tensor generalized Killing equations ensure that

$$Q_K = X_\mu u^\mu \,,$$

$$Q_{SK} = K_{\mu_1 \cdots \mu_m} u^{\mu_1} \cdots u^{\mu_m} \,,$$

are constants along γ.
For a conformal Killing vector or a conformal Stackel-Killing tensor the quantities

$$Q_{CK} = Z_\mu u^\mu \,,$$

$$Q_{CSK} = A_{\mu\nu} u^\mu u^\nu \,,$$

are conserved along a null geodesic γ

$$u^\mu u_\mu = 0 \,.$$

Quantum anomalies (1)

In the quantum case, the momentum operator is given by ∇^μ and the Hamiltonian operator for a free scalar particle is the covariant Laplacian acting on scalars

$$\mathcal{H} = \Box = \nabla_\mu g^{\mu\nu} \nabla_\nu = \nabla_\mu \nabla^\mu$$

For a CK vector we define the conserved operator in the quantized system as

$$\mathcal{Q}_{CK} = Z^\mu \nabla_\mu \, .$$

In order to identify a quantum gravitational anomaly we shall evaluate the commutator $[\Box, \mathcal{Q}_{CK}]\Phi$, for $\Phi \in \mathcal{C}^\infty(M)$ solutions of the Klein-Gordon equation.

Quantum anomalies (1)

Explicit evaluation of the commutator:

$$[\mathcal{H}, \mathcal{Q}_{CK}] = \frac{2-n}{n} Z_{\sigma}{}^{;\sigma\mu}\nabla_{\mu} + \frac{2}{n} Z^{\sigma}{}_{;\sigma}\Box \,.$$

In the case of ordinary K vectors, the r. h. s. of this commutator vanishes and there are no quantum gravitational anomalies. However for CK vectors, the situation is quite different. Even if we evaluate the r. h. s. of the commutator on solutions of the massless Klein-Gordon equation, $\Box\Phi(x) = 0$,the term $Z_{\sigma}{}^{;\sigma\mu}\nabla_{\mu}$ survives. Only in a very special case, when by chance this term vanishes, the anomalies do not appear.

Quantum anomalies (2)

Quantum analog of conserved quantities for Killing tensors $K^{\mu\nu}$

$$\mathcal{Q}_{SK} = \nabla_\mu K^{\mu\nu} \nabla_\nu \,.$$

Similar form for $\mathcal{Q}_{CSK}$ constructed from a conformal Stackel-Killing tensor.
Evaluation of the commutator (Carter 1977)

$$\begin{aligned}
&[\Box, \mathcal{Q}_{SK}] = 2(\nabla^{(\sigma} K^{\mu\nu)} \nabla_\sigma \nabla_\mu \nabla_\nu \\
&+3\nabla_\kappa \left(\nabla^{(\sigma} K^{\kappa\nu)} \right) \nabla_\nu \nabla_\sigma \\
&+\left\{ -\frac{4}{3} \nabla_\sigma \left(R_\kappa{}^{[\sigma} K^{\nu]\kappa} \right) \right. \\
&\left. + \nabla_\sigma \left(\frac{1}{2} g_{\kappa\lambda} (\nabla^\sigma \nabla^{(\kappa} K^{\lambda\nu)} - \nabla^\nu \nabla^{(\kappa} K^{\sigma\lambda)}) + \nabla_\mu \nabla^{(\sigma} K^{\mu\nu)} \right) \right\} \nabla_\nu
\end{aligned}$$

Quantum anomalies (3)

In case of Stackel-Killing tensors the commutator simplifies:

$$[\Box, \mathcal{Q}_{SK}] = -\frac{4}{3}\nabla_\mu (R_\lambda{}^{[\mu} K^{\nu]\lambda})\nabla_\nu\,.$$

There are a few notable conditions for which the commutator vanishes, i.e. No anomalies:

- Space is Ricci flat, i. e. $R_{\mu\nu} = 0$
- Space is Einstein, i. e. $R_{\mu\nu} \propto g_{\mu\nu}$
- Stackel-Killing tensors associated to Killing-Yano tensors of rank 2:

$$K_{\mu\nu} = \omega_{\mu\lambda}\omega^\lambda{}_\nu.$$

Quantum anomalies (4)

In case of conformal Stackel-Killing tensors there are quantum gravitational anomalies. Even if we evaluate the commutator for a conformal Stackel-Killing tensor associated to a conformal Killing-Yano tensor

$$A_{\mu\nu} = f_{\mu\lambda} f^{\lambda}_{\nu}.$$

the commutator does not vanish

$$\begin{aligned}
[\Box, \mathcal{Q}_{CSK}] &= \frac{4}{n-1} f_\lambda f^{\lambda\sigma} \nabla_\sigma \Box \\
&+ \frac{4}{n-1} \left(\nabla^\nu f_\lambda f^{\lambda\sigma} \right) \nabla_\nu \nabla_\sigma + \frac{2}{n-1} \left(\nabla_\sigma f_\lambda f^{\lambda\sigma} \right) \Box \\
&+ \left\{ \frac{14-n}{6(n-1)} f_\lambda f^{\lambda\sigma} R_\sigma^{\ \nu} - \frac{4(n-2)}{3(n-1)} \nabla_\sigma \left(f^{(\sigma;\lambda)} f^{\nu}_{\ \lambda} - f^{(\nu;\lambda)} f^{\sigma}_{\ \lambda} \right) \right. \\
&+ \left. \frac{n+4}{3(n-1)} \left(\Box f_\lambda f^{\lambda\sigma} \right) - \frac{n-2}{3(n-1)} \left(\nabla^{(\sigma} \nabla^{\nu)} f_\lambda f^{\lambda}_{\ \sigma} \right) \right\} \nabla_\nu
\end{aligned}$$

Gauge covariant approach (1)

Classical dynamics of a point charge q of mass M in the external Abelian gauge field A_i and a scalar potential $V(x^i)$

$$H = \frac{1}{2M} g^{ij}(p_i - qA_i)(p_j - qA_j) + V \,.$$

Hamilton equations of motion are not manifestly gauge covariant.

Gauge covariant approach (2)

Gauge covariant formulation (van Holten 2007)

$$\mathbf{\Pi} = \mathbf{p} - q\mathbf{A} = M\dot{\mathbf{x}}\,.$$

Hamiltonian becomes

$$H = \frac{1}{2M} g^{ij} \Pi_i \Pi_j + V\,,$$

Covariant Poisson brackets

$$\{f, g\} = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial \Pi_i} - \frac{\partial f}{\partial \Pi_i}\frac{\partial g}{\partial x^i} + qF_{ij}\frac{\partial f}{\partial \Pi_i}\frac{\partial g}{\partial \Pi_j}\,.$$

where $F_{ij} = A_{j;i} - A_{j;i}$ is the field strength.

Gauge covariant approach (3)

Fundamental Poisson brackets

$$\{x^i, x^j\} = 0\,,\ \{x^i, \Pi_j\} = \delta^i_j\,,\ \{\Pi_i, \Pi_j\} = qF_{ij}\,,$$

Momenta $\boldsymbol{\Pi}$ are not canonical.
Hamilton's equations:

$$\dot{x}^i = \{x^i, H\} = \frac{1}{M} g^{ij} \Pi_j\,,$$

$$\dot{\Pi}_i = \{\Pi_i, H\} = qF_{ij}\dot{x}^j - V_{,i}\,.$$

Gauge covariant approach (4)

Conserved quantities of motion in terms of phase-space variables (x^i, Π_i)

$$K = K_0 + \sum_{n=1}^{p} \frac{1}{n!} K_n^{i_1 \cdots i_n}(x) \cdots \Pi_{i_1} \Pi_{i_n} \,,$$

Bracket

$$\{K, H\} = 0 \,.$$

vanishes.

Gauge covariant approach (5)

Series of constraints:

$$K_1^i V_{,i} = 0\,,$$

$$K_{0,i} + qF_{ji} K_1^j = M K_{2i}^{j} V_{,j}\,,$$

$$K_n^{(i_1 \cdots i_n ; i_{n+1})} + qF_j{}^{(i_{n+1}} K_{n+1}^{i_1 \cdots i_n)j} = \frac{M}{(n+1)} K_{n+2}^{i_1 \cdots i_{n+1} j} V_{,j}$$

$$for \;\; n = 1\,, \cdots (p-2)\,,$$

$$K_{p-1}^{(i_1 \cdots i_{p-1} ; i_p)} + qF_j{}^{(i_p} K_p^{i_1 \cdots i_{p-1})j} = 0\,,$$

$$K_p^{(i_1 \cdots i_p ; i_{p+1})} = 0\,.$$

Gauge covariant approach (6)

Role of Killing-Yano tensors (1)

In pseudo-classical spinning particles models the condition of the electromagnetic field $F_{\mu\nu}$ to maintain the non generic supersymmetry associated with a KY tensor ω of rank p is

$$F_{\nu[\mu_p}\omega_{\mu_1\cdots\mu_{p-1}]}{}^{\nu} = 0\,,$$

Consequences of this condition for the series of constraints
Assume that the Stäckel-Killing tensor $K_{2\mu\nu}$ is associated with a Killing-Yano tensor $\omega_{\mu\nu}$

$$K_{2\mu\nu} = \omega_{\mu\lambda}\omega^{\lambda}{}_{\nu}\,.$$

In this case, condition for the electromagnetic field $F_{\mu\nu}$ reads

$$F_{\lambda[\mu}\omega_{\nu]}{}^{\lambda} = 0\,.$$

Gauge covariant approach (7)

Role of Killing-Yano tensors (2)

We get

$$F_j{}^{i_2} K_2^{i_1 j} = 0\,.$$

Therefore Killing-Yano tensors prove to produce significant simplifications in the series of constraints for the higher order integrals of motion.

Gauge covariant approach (8)

Examples (1)

Consider $\mathcal{M}$ to be a 3-dimensional Euclidean space $\mathbb{E}^3$

We investigate the constant of motion in a Kepler-Coulomb potential adding different types of electric and magnetic fields

We consider the motion of a point charge q of mass M in the Coulomb potential Q/r produce by a charge Q when some external electric or magnetic fields are also present.

Non relativistic Kepler-Coulomb problem admits two vector constants of motion

- angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{\Pi} \,,$$

- Runge-Lenz vector

$$\mathbf{K} = \mathbf{\Pi} \times \mathbf{L} + MqQ\frac{\mathbf{r}}{r} \,.$$

Gauge covariant approach (9)

Examples I. Constant electric field (I.1)

Electric charge q moves in the Coulomb potential with a constant electric field $\mathbf{E}$ present.

Hamiltonian:

$$H = \frac{1}{2M}\mathbf{\Pi}^2 + q\frac{Q}{r} - q\mathbf{E}\cdot\mathbf{r}\,,$$

with $\mathbf{\Pi} = M\dot{\mathbf{r}}$ in spherical coordinates of $\mathbb{E}^3$.

Gauge covariant approach (10)

Examples I. Constant electric field (I.2)

Looking for a constant of motion of the form

$$K = K_0 + K_{1i}\Pi_i + \frac{1}{2}K_{2ij}\Pi_i\Pi_j\,.$$

Components K_{2ij} are Stäckel-Killing tensors, of rank $p = 2$

$$K_{2ij} = 2\delta_{ij}\mathbf{n}\cdot\mathbf{r} - (n_i r_j + n_j n_i)\,,$$

written in spherical coordinates with $\mathbf{n}$ an arbitrary constant vector.

Choose $\mathbf{n}$ along $\mathbf{E}$

Gauge covariant approach (11)

Examples I. Constant electric field (I.3)

Solution of the series of constraints for a first integral of motion

$$K_0 = \frac{MqQ}{r}\mathbf{E}\cdot\mathbf{r} - \frac{Mq}{2}\mathbf{E}\cdot\left[\mathbf{r}\times(\mathbf{r}\times\mathbf{E})\right].$$

$$\mathbf{K}_1 = \mathbf{r}\times\mathbf{E}\,,$$

modulo an arbitrary constant factor. This vector $\mathbf{K}_1$ contribute to a conserved quantity with a term proportional to the angular momentum $\mathbf{L}$ along the direction of the electric field $\mathbf{E}$. In conclusion, when a uniform constant electric field is present, the KC system admits two constants of motion $\mathbf{L}\cdot\mathbf{E}$ and $\mathbf{C}\cdot\mathbf{E}$ where $\mathbf{C}$ is a generalization of the Runge-Lenz vector

$$\mathbf{C} = \mathbf{K} - \frac{Mq}{2}\mathbf{r}\times(\mathbf{r}\times\mathbf{E})\,.$$

Gauge covariant approach (12)

Examples II. Spherically symmetric magnetic field (II.1)

Spherically symmetric magnetic field

$$\mathbf{B} = f(r)\mathbf{r}\,,$$

$$F_{ij} = \epsilon_{ijk} B_k = \epsilon_{ijk} r_k f(r)\,,$$

+ Coulomb potential acting on a electric charge q.
Start with a Stäckel-Killing K_{2ij} of rank 2 as in in the previous example.
From the hierarchy of constraints we get

$$K_{1i} = q\left[\int rf(r)dr\right](\mathbf{n}\times\mathbf{r})_i\,,$$

Gauge covariant approach (13)

Examples II. Spherically symmetric magnetic field (II.2)

Equation for K_0 can be solely solved making choice of a definite form for the function $f(r)$

$$f(r) = \frac{g}{r^{5/2}}\,,$$

with g a constant connected with the strength of the magnetic field.

With this special form of the function $f(r)$ we get

$$K_0 = \left[\frac{MqQ}{r} - \frac{2g^2q^2}{r}\right](\mathbf{n}\cdot\mathbf{r})\,,$$

and

$$K_{1i} = -\frac{2gq}{r^{1/2}}(\mathbf{r}\times\mathbf{n})_i\,.$$

Gauge covariant approach (14)

Examples II. Spherically symmetric magnetic field (II.3)

Collecting the terms K_0, K_{1i}, K_{2ij} the constant of motion becomes

$$K = \mathbf{n} \cdot \left(\mathbf{K} + \frac{2gq}{r^{1/2}} \mathbf{L} - 2g^2 q^2 \frac{\mathbf{r}}{r} \right) ,$$

with $\mathbf{n}$ an arbitrary constant unit vector and $\mathbf{K}, \mathbf{L}$ as in the pure Coulomb problem.

Gauge covariant approach (15)

Examples III. Magnetic field along a fixed direction (III.1)

Magnetic field along a fixed direction $\mathbf{n}$

$$\mathbf{B} = B(\mathbf{r} \cdot \mathbf{n})\mathbf{n}\,,$$

where, for the beginning, $B(\mathbf{r} \cdot \mathbf{n})$ is an arbitrary function.
Again start with a Stäckel-Killing K_{2ij} of rank 2 and we get

$$K_{1i} = q\left[\int rB(\mathbf{r} \cdot \mathbf{n})d(\mathbf{r} \cdot \mathbf{n})\right](\mathbf{r} \times \mathbf{n})_i\,.$$

Equation for K_0 proves to be solvable for a particular form of the magnetic field

$$\mathbf{B} = \frac{\alpha}{\sqrt{\alpha\mathbf{r} \cdot \mathbf{n} + \beta}}\,\mathbf{n}\,,$$

with α, β two arbitrary constants.

Gauge covariant approach (16)

Examples III. Magnetic field along a fixed direction (III.2)

Finally we get for K_0 and K_{1i}

$$K_0 = \frac{MqQ}{r}(\mathbf{r}\cdot\mathbf{n}) + \alpha q^2(\mathbf{r}\times\mathbf{n})^2\,,$$

$$K_{1i} = -2q\sqrt{\alpha\mathbf{r}\cdot\mathbf{n}+\beta}\,(\mathbf{r}\times\mathbf{n})_i\,.$$

Constant of motion for this configuration of the magnetic field superposed on the Coulomb potential becomes:

$$K = \mathbf{n}\cdot\left[\mathbf{K} + 2q\sqrt{\alpha\mathbf{r}\cdot\mathbf{n}+\beta}\ \ \mathbf{L}\right] + \alpha q^2(\mathbf{r}\times\mathbf{n})^2\,.$$

As in the previous example the angular momentum $\mathbf{L}$ is no longer conserved, forming part of the constant of motion K **.**

Outlook

- Non-Abelian dynamics
- Spaces with skew-symmetric torsion
- Higher order Killing tensors (rank ≥ 3)
- .....

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