Conformal Einstein Perfect Fluid Spacetimes

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References:

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  (thesis supervised by Brian Edgar and Magnus Herberthson)

$$\hat{g}_{ij} = e^{2\phi} g_{ij} \Longrightarrow e^{2\phi} \hat{R}^{ij}{}_{kl} = R^{ij}{}_{kl} - 4 Y^{[i}_{[k} \delta^{j]}_{l]} \tag{1}$$

with

$$Y_{ij} = \phi_{,i;j} - \phi_{,i}\phi_{,j} + \frac{1}{2} g_{ij} g^{mn} \phi_{,m}\phi_{,n}. \tag{2}$$

$(\mathcal{M}, g)$ conformal Einstein perfect fluid:

$$\begin{cases} \hat{R}_{ij} - \frac{1}{2}\hat{R}\hat{g}_{ij} + \Lambda \hat{g}_{ij} = 0 \\ R_{ij} - \frac{1}{2} R g_{ij} = (m+p) u_i u_j + p g_{ij} \end{cases} \tag{3}$$

$$\Longrightarrow Y_{ij} = \frac{1}{2}(m+p) u_i u_j + \frac{1}{6}(m - \Lambda e^{2\phi}) g_{ij}. \tag{4}$$

Introduction

1986, 1987:

- $\Lambda = 0$, $\nabla\phi \parallel \boldsymbol{u}$
- $\Lambda = 0$, shear-free
- $\Lambda = 0$, vorticity-free
- $\Lambda = 0$, inheriting G_2
- $\Lambda = 0$, barotropic equation of state and $\nabla\phi \perp \boldsymbol{u}$

(Geroch, Held and Penrose, JMP 1973)

$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4 \equiv \boldsymbol{m}, \bar{\boldsymbol{m}}, \boldsymbol{\ell}, \boldsymbol{k}$ with $\boldsymbol{k} \cdot \boldsymbol{\ell} = -1, \boldsymbol{m} \cdot \bar{\boldsymbol{m}} = 1$

Under boosts

$$\boldsymbol{k} \to A\boldsymbol{k},\ \boldsymbol{\ell} \to A^{-1}\boldsymbol{\ell}$$

and spatial rotations

$$\boldsymbol{m} \to e^{i\theta}\boldsymbol{m}$$

well-weighted variables η of weight $[p, q]$ transform as

$$\eta \to A^{\frac{p+q}{2}} e^{i\frac{p-q}{2}\theta}\eta$$

(η has *boost-weight* $= \frac{p+q}{2}$ and *spin-weight* $= \frac{p-q}{2}$).

Basic variables:

$$\kappa = \Gamma_{414}, \quad \tau = \Gamma_{413}, \quad \sigma = \Gamma_{411}, \quad \rho = \Gamma_{412},$$
$$\nu = \Gamma_{233}, \quad \pi = \Gamma_{234}, \quad \lambda = \Gamma_{232}, \quad \mu = \Gamma_{231},$$

$(\Gamma_{abc} = -\Gamma_{bac} \equiv \boldsymbol{e}_a \nabla_c(\boldsymbol{e}_b))$,

$$\Phi_{00}, \Phi_{22}, \Phi_{01}, \Phi_{12}, \Phi_{02}, \Phi_{11},$$
$$R, \Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4.$$

$\alpha, \beta, \epsilon, \gamma$ get absorbed in $Þ, Þ', ð, ð'$:

$$\begin{aligned} Þ\eta &= (D - p\epsilon - q\bar{\epsilon})\eta \\ Þ'\eta &= (\Delta - p\gamma - q\bar{\gamma})\eta \\ ð\eta &= (\delta - p\beta - q\bar{\alpha})\eta \\ ð'\eta &= (\bar{\delta} - p\alpha - q\bar{\beta})\eta \end{aligned}$$

GHP formalism

Symmetry transformations:

- complex conjugation
- prime transformation:

$$\boldsymbol{k} \leftrightarrow \boldsymbol{\ell}, \boldsymbol{m} \leftrightarrow \bar{\boldsymbol{m}},$$

$$\kappa \leftrightarrow -\nu, \quad \tau \leftrightarrow -\pi, \quad \sigma \leftrightarrow -\lambda, \quad \rho \leftrightarrow -\mu,$$
$$\Phi_{ij} \leftrightarrow \Phi_{2-i\,2-j}, \quad \Psi_i \leftrightarrow \Psi_{4-i}$$

- Sachs transformation:

$$\boldsymbol{k} \rightarrow \boldsymbol{m}, \boldsymbol{\ell} \rightarrow \bar{\boldsymbol{m}}, \boldsymbol{m} \rightarrow \bar{\boldsymbol{k}}, \bar{\boldsymbol{m}} \rightarrow \boldsymbol{\ell}$$

$$\rho^* = \tau, \tau^* = -\rho, \rho'^* = -\tau', \tau'^* = \rho'$$

Basic equations:

- 12 complex Ricci equations

$$
\begin{aligned}
Þ\tau - Þ'\kappa &= (\tau - \bar{\tau}')\rho + (\bar{\tau} - \tau')\sigma + \Phi_{01} + \Psi_1, \\
ð\rho - ð'\sigma &= (\rho - \bar{\rho})\tau + (\bar{\rho}' - \rho')\kappa + \Phi_{01} - \Psi_1, \\
Þ\sigma - ð\kappa &= (\rho + \bar{\rho})\sigma - (\tau + \bar{\tau}')\kappa + \Psi_0, \\
Þ\rho - ð'\kappa &= \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \kappa\tau' + \Phi_{00}, \\
Þ'\sigma - ð\tau &= \sigma\rho' - \bar{\lambda}\rho - \tau^2 + \kappa\bar{\nu} - \Phi_{02}, \\
Þ'\rho - ð'\tau &= \rho\bar{\rho}' - \lambda\sigma - \tau\bar{\tau} + \kappa\nu - \Psi_2 - \frac{1}{12}R
\end{aligned}
$$

- 9 complex + 2 real Bianchi equations
- commutator relations

Construct $(\boldsymbol{m}, \bar{\boldsymbol{m}}, \boldsymbol{k}', \boldsymbol{k})$ such that $\mathbb{R}\{\boldsymbol{u}, \nabla\phi\} = \mathbb{R}\{\boldsymbol{k}, \boldsymbol{k}'\}$:

- $\eth\phi \; (= \eth'\phi) = 0$
- $\boldsymbol{u} = \frac{1}{\sqrt{2}}(Q\boldsymbol{k}' + Q'\boldsymbol{k}) \quad (w(Q) = [1,1],\; Q' = 1/Q)$

$$\Downarrow$$

$$\begin{cases} Q'^2\Phi_{00} = Q^2\Phi_{22} = 2\Phi_{11} = \frac{1}{4}(m+p) \\ R = m - 3p \\ \Phi_{01} = \Phi_{02} = \Phi_{12} = 0 \end{cases} \tag{5}$$

$$\left\{ \begin{array}{l} \text{Þ}^2\phi = (\text{Þ}\phi)^2 + \frac{1}{4}(m+p)Q^2 \\ \text{Þ}'\text{Þ}\phi = \frac{1}{4}(m+3p) + \frac{1}{6}\Lambda e^{2\phi} \end{array} \right. \tag{6}$$

$$\left\{ \begin{array}{l} \bar{\sigma}\text{Þ}'\phi + \sigma'\text{Þ}\phi = 0 \\ \rho\text{Þ}'\phi + \bar{\rho}'\text{Þ}\phi = (\text{Þ}\phi)(\text{Þ}'\phi) + \frac{1}{6}(m - \Lambda e^{2\phi}) \\ \kappa\text{Þ}'\phi - \bar{\pi}\text{Þ}\phi = 0 \end{array} \right. \tag{7}$$

Some natural sub-cases

$$N := Q'^2 \text{Þ}\phi / \text{Þ}'\phi \quad (w(N) = [0,0], N' = 1/N)$$

I. $N = 1$ ($\boldsymbol{u}/parallel \nabla\phi$):

$$\text{FLRW}$$

II. $(N+3)(N'+3) = 0$:

$$\begin{cases} \sigma_{ab} = \omega_{ab} = 0 \\ \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \\ \Psi_2 - \bar{\Psi}_2 = 0 \\ \Lambda < 0 \\ m < 0 \end{cases}$$

III. $N^2 + 6N + 1 = 0$: idem

$$(\alpha_1 N + \alpha_2)m^2 \; + \; (\dots N + \dots)m \; + \; \dots N + \dots = 0$$

$$(N = -3 \pm 2\sqrt{2}, \;\; \alpha_1, \alpha_2, \dots \in \mathbb{N}).$$

$$\text{discriminant} = \zeta_1 + \zeta_2 N$$

a surprise on the way ...

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$$\zeta_1 = 1357297565629151284215396601095623063987021170856423988 1$$
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$$3 - 2\sqrt{2} - \frac{\zeta_1}{\zeta_2} \approx 0.28\ 10^{-112}$$

a surprise on the way . . .

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$$(p, q \in \mathbb{N}:\ \ \tfrac{p}{q} = (3 - \tfrac{\zeta_1}{\zeta_2})/2 \ \ \text{satisfy}\ p - q\sqrt{2} = (1 - \sqrt{2})^{148}).$$

Shear-free case (and $N \neq 1, -3, -1/3, -3 \pm 2\sqrt{2}$)

$$\sigma_{ab} = 0 \Longrightarrow \begin{cases} \boldsymbol{u} \perp \nabla\phi \; (N = -1) \\ \theta = 0 \end{cases}$$

Not type O $\Longrightarrow \kappa = 0, \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0.$

1. $\text{Þ}\phi = \rho$: LRS IIc special static fluid spheres (or generalisations):

$$\begin{cases} \delta \equiv \bar{\delta} \equiv 0 \\ D\phi = -\Delta\phi = k \; (\text{constant}) \\ (D - \Delta)p = \frac{m+p}{2k}(p + 2\Lambda e^{2\phi} - 2k^2) \\ \Psi_2 = -\frac{2}{3}\Lambda e^{2\phi} \\ \Lambda < 0 \\ m = -2\Lambda e^{2\phi} - 6k^2 \end{cases} \qquad (8)$$

2. $\text{Þ}\phi \neq \rho$: $\Lambda \neq 0$-generalisations of 1986-results:
   a) $\bar{\rho} \neq \rho$: LRS I, Cahen-Defrise-transforms
   b) $\bar{\rho} = \rho$: LRS IIc and IIb, " " "

Vorticity-free case (and $N \neq 1, -3, -1/3, -3 \pm 2\sqrt{2}$)

$\omega_{ab} = 0$ (and $\sigma_{ab} \neq 0$) $\Longrightarrow \Lambda \neq 0$-generalisations of the 1986-results :

1. $\boldsymbol{u} \perp \nabla\phi$:

$$G_3 \; VI_0 \text{ on } T_3$$

2. $\boldsymbol{u} \not\perp \nabla\phi$:

   special (pseudo-)spherically or plane symmetric expanding fluids

To do

- $\boldsymbol{u} \perp \nabla\phi$: promising ...
- general case: ???