Geometric invariants for initial data sets: analysis, exact solutions, computer algebra, numerics

Juan A. Valiente Kroon

School of Mathematical Sciences
Queen Mary, University of London

j.a.valiente-kroon@qmul.ac.uk,

ERE2010, Granada, September 2010

Outline

1. Motivation
2. Invariant characterisations of Kerr
3. Invariant characterisation of Kerr initial data

Characterising slices in the Kerr spacetime?

Objective of this talk:

- Illustrate the interaction between mathematical Relativity (Exact Solutions, Analysis) and computer methods (Computer Algebra, Numerics) through an example of my own research —done in collaboration with T. Bäckdahl and A. García Parrado.

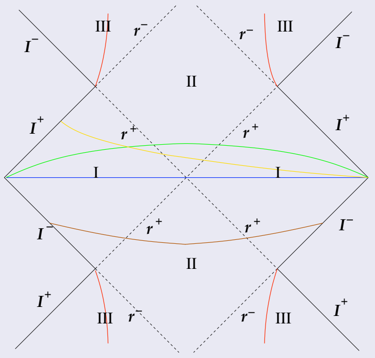
Characterising slices in the Kerr spacetime?

Objective of this talk:

- Illustrate the interaction between mathematical Relativity (Exact Solutions, Analysis) and computer methods (Computer Algebra, Numerics) through an example of my own research —done in collaboration with T. Bäckdahl and A. García Parrado.

A problem:

- Given a solution to the Einstein vacuum constraints, $(\mathcal{S}, h_{ij}, K_{ij})$, under which conditions is it a slice of the Kerr spacetime?



Difficulties in the interaction

A further objective of the talk:

Exemplify the difficulties in the interaction...

Difficulties in the interaction

A further objective of the talk:

Exemplify the difficulties in the interaction...

Some examples of this in this talk:

- Different languages and methods —e.g. use of spinors in the analysis.
- Different objectives:
  - Mathematical relativists enjoy problems with a lot of structure and want to prove theorems.
  - Numerical relativists want (not too computationally taxing) tools to analyse their numerically constructed solutions.

Motivations to study characterisations of Kerr data

Numerical Relativity:

In numerical simulations of dynamical black hole spacetimes (which make use of 3+1 formulations of GR), one expects that the final state will be close to Kerr/Schwarzschild.

- Can one make invariant statements about this?

Motivations to study characterisations of Kerr data

Numerical Relativity:

In numerical simulations of dynamical black hole spacetimes (which make use of 3+1 formulations of GR), one expects that the final state will be close to Kerr/Schwarzschild.

- Can one make invariant statements about this?

Non-linear stability of Kerr:

One expects that a dynamical spacetime which is the development of data close to Kerr data will have the same asymptotic structure as Kerr.

- What does it mean to be close to Kerr? (either at the data or spacetime level)

Motivations to study characterisations of Kerr data

Numerical Relativity:

In numerical simulations of dynamical black hole spacetimes (which make use of 3+1 formulations of GR), one expects that the final state will be close to Kerr/Schwarzschild.

- Can one make invariant statements about this?

Non-linear stability of Kerr:

One expects that a dynamical spacetime which is the development of data close to Kerr data will have the same asymptotic structure as Kerr.

- What does it mean to be close to Kerr? (either at the data or spacetime level)

Construction of initial data sets:

There is a number of conjectures about the existence/non-existence of certain types of hypersurfaces in Kerr/Schwarzschild. An invariant characterisation of data could be of use to construct the required data or to conclude that they do not exist.

A simple example: slices of Minkowski spacetime

Local solution to the problem of characterisation of initial data:

- The pair (h_{ij}, K_{ij}) of symmetric tensors corresponds (locally) to the first and second fundamental form of a slice $\mathcal{S}$ in Minkowski spacetime if and only if

$$D_{[i}K_{j]l} = 0,$$
$$r_{ijkl} = -2K_{k[i}K_{j]l}.$$

A simple example: slices of Minkowski spacetime

Local solution to the problem of characterisation of initial data:

- The pair (h_{ij}, K_{ij}) of symmetric tensors corresponds (locally) to the first and second fundamental form of a slice $\mathcal{S}$ in Minkowski spacetime if and only if

$$D_{[i}K_{j]l} = 0,$$
$$r_{ijkl} = -2K_{k[i}K_{j]l}.$$

A global characterisation (Schoen & Yau):

- The pair, (h_{ij}, K_{ij}), of smooth asymptotically Euclidean symmetric tensors corresponds (locally) to the first and second fundamental form of a slice $\mathcal{S}$ in Minkowski spacetime if and only if its ADM mass is zero.

Outline

1. Motivation
2. Invariant characterisations of Kerr
3. Invariant characterisation of Kerr initial data

The input from the theory of Exact Solutions

First task:

- Find a suitable invariant (spacetime) characterisation of the Kerr solution.

The input from the theory of Exact Solutions

First task:

- Find a suitable invariant (spacetime) characterisation of the Kerr solution.

Observation:

- There are a number of such characterisations in the literature.

The input from the theory of Exact Solutions

First task:

- Find a suitable invariant (spacetime) characterisation of the Kerr solution.

Observation:

- There are a number of such characterisations in the literature.

Further requirements:

- Should be amenable to a 3+1 decomposition.
- Should be as simple as possible (this may require including global conditions).
- Should be amenable to analytic methods (purely algebraic conditions are not that ideal).

Some invariant characterisations of Kerr

Some characterisations of the Kerr spacetime:

- There exist already some local characterisations of the Kerr spacetime and of Kerr data:
  - A characterisation in terms of the Mars-Simon tensor (Mars, 1999-2000).
  - A characterisation in terms of concomitants of the Weyl tensor (Ferrando & Sáez, 2009).
  - A characterisation in terms of Killing spinors
    - Penrose, Hugston, Sommers, Walker, Floyd (1970's); A. García-Parrado, T. Bäckdahl, JAVK (2007-2010).

Some invariant characterisations of Kerr

Some characterisations of the Kerr spacetime:

- There exist already some local characterisations of the Kerr spacetime and of Kerr data:
  - A characterisation in terms of the Mars-Simon tensor (Mars, 1999-2000).
  - A characterisation in terms of concomitants of the Weyl tensor (Ferrando & Sáez, 2009).
  - A characterisation in terms of Killing spinors
    - Penrose, Hugston, Sommers, Walker, Floyd (1970's); A. García-Parrado, T. Bäckdahl, JAVK (2007-2010).

Observation:

- In practice a combination of all of the above is what works best!

Killing spinors as a characterisation of type D spacetimes

Observation:

- The Weyl tensor of the Kerr spacetime is (everywhere) of Petrov type D.

Killing spinors as a characterisation of type D spacetimes

Observation:

- The Weyl tensor of the Kerr spacetime is (everywhere) of Petrov type D.

Rationale:

- Characterisations in terms of the algebraic conditions determining the Petrov type (e.g. type D) are cumbersome.
- Can one find a way of using the Petrov type in an indirect manner?

Killing spinors as a characterisation of type D spacetimes

Observation:

- The Weyl tensor of the Kerr spacetime is (everywhere) of Petrov type D.

Rationale:

- Characterisations in terms of the algebraic conditions determining the Petrov type (e.g. type D) are cumbersome.
- Can one find a way of using the Petrov type in an indirect manner?

Killing spinors:

- A Killing spinor is a totally symmetric spinor $\kappa_{AB} = \kappa_{(AB)}$ satisfying

$$\nabla_{Q'(Q}\kappa_{AB)} = 0.$$

Observation:

Equivalent to have a conformal Killing-Yano tensor:

$$\nabla_{(\mu}K_{\nu)\lambda} = 0, \quad K_{\mu\nu} = -K_{\nu\mu}.$$

A characterisation of Kerr:

Theorem

Let $(\mathcal{M}, g_{\mu\nu})$ *be a smooth vacuum spacetime such that*

$$\Psi_{ABCD} \neq 0, \quad \Psi_{ABCD}\Psi^{ABCD} \neq 0.$$

The spacetime is locally isometric to the Kerr spacetime if and only if the following conditions are satisfied:

- *there exists a Killing spinor* κ_{AB} *such that* $\xi_{AA'} \equiv \nabla^Q{}_{A'}\kappa_{AQ}$ *is real.*
- *the spacetime has a stationary asymptotically flat ends with non-vanishing mass in which* $\xi_{AA'}$ *tends to a time translation.*

A characterisation of Kerr:

Theorem

Let $(\mathcal{M}, g_{\mu\nu})$ be a smooth vacuum spacetime such that

$$\Psi_{ABCD} \neq 0, \quad \Psi_{ABCD}\Psi^{ABCD} \neq 0.$$

The spacetime is locally isometric to the Kerr spacetime if and only if the following conditions are satisfied:

- *there exists a Killing spinor κ_{AB} such that $\xi_{AA'} \equiv \nabla^{Q}{}_{A'}\kappa_{AQ}$ is real.*
- *the spacetime has a stationary asymptotically flat ends with non-vanishing mass in which $\xi_{AA'}$ tends to a time translation.*

Observation:

- This is essentially a reformulation of a characterisation given by M. Mars (2000).

Outline

1. Motivation
2. Invariant characterisations of Kerr
3. Invariant characterisation of Kerr initial data

A split of the Killing spinor equation

Second task:

- Want to rewrite the theorem for the invariant characterisation of the Kerr spacetime in terms of conditions on initial data on a spacelike hypersurface.
  - For this, need to encode the existence of a Killing spinor in the spacetime at the level of initial data.
  - Need to perform a split of the spacetime Killing spinor equation.

A split of the Killing spinor equation

Second task:

- Want to rewrite the theorem for the invariant characterisation of the Kerr spacetime in terms of conditions on initial data on a spacelike hypersurface.
  - For this, need to encode the existence of a Killing spinor in the spacetime at the level of initial data.
  - Need to perform a split of the spacetime Killing spinor equation.

Implementation of the idea:

- One requires to perform lenghty spinorial computations.
  - These were carried out in J.M. Martín Garcia's suite `xAct` for `Mathematica`.

$$
\begin{aligned}
\nabla^{AB}\nabla_{AB}\xi = {} & -\tfrac{1}{6}K^2\xi - \tfrac{1}{2}\Omega^{ABCD}\Omega_{ABCD}\xi + 3\Psi_A{}^{CDF}\Omega_{BCDF}\kappa^{AB} + \xi_{AB}\nabla^{AB}K \\
& + \tfrac{3}{4}\hat{\Psi}^{ABCD}\xi_{ABCD} - \tfrac{9}{4}\Psi^{ABCD}\xi_{ABCD} + 2K\Omega^{ABCD}\xi_{ABCD} \\
& - \tfrac{15}{4}\Omega^{ABFH}\Omega^{CD}{}_{FH}\xi_{ABCD} + \tfrac{9}{2}\Omega^{ABCD}\nabla^F{}_D\xi_{ABCF} \\
& + \tfrac{3}{2}\nabla^{AB}\nabla^{CD}\xi_{ABCD}, \qquad (19a)\\
\nabla^C{}_{(A}\nabla_{B)C}\xi = {} & \tfrac{1}{2}\Omega_{ABCD}\nabla^{CD}\xi - \tfrac{1}{3}K\nabla_{AB}\xi, \qquad (19b)\\
\nabla_{(AB}\nabla_{CD)}\xi = {} & -4K\Psi_{(ABC}{}^E\kappa_{D)E} + \tfrac{1}{2}\hat{\Psi}_{ABCD}\xi - \tfrac{5}{2}\Psi_{ABCD}\xi - \tfrac{2}{3}\hat{\Psi}_{(ABC}{}^E\xi_{D)E} \\
& - \tfrac{10}{3}\Psi_{(ABC}{}^E\xi_{D)E} + \Omega_{ABCDEL}\xi^{EL} + \tfrac{4}{3}K^2\xi_{ABCD} + 3\Omega_{EFL(ABC}\xi_{D)}{}^{ELF} \\
& + 3\Psi_{(AB}{}^{EL}\xi_{CD)EL} - \tfrac{3}{2}\xi_{(A}{}^{ELF}\Omega_{BCD)}{}^H\Omega_{ELFH} - 3\Psi_{EL(A}{}^F\kappa^{EL}\Omega_{BCD)F} \\
& - \xi^{EL}\Omega_{ELF(A}\Omega_{BCD)}{}^F + \tfrac{2}{3}K\xi_{(A}{}^E\Omega_{BCD)E} + \tfrac{1}{2}\xi^{ELFH}\Omega_{EL(AB}\Omega_{CD)FH} \\
& - 3\Psi_{E(B}{}^{LF}\kappa_A{}^E\Omega_{CD)LF} - 3\Psi_{E(AB}{}^F\kappa^{EL}\Omega_{CD)LF} - \Omega_{ELF(B}\xi_A{}^E\Omega_{CD)}{}^{LF} \\
& - 4K\xi_{(AB}{}^{EL}\Omega_{CD)EL} - \tfrac{1}{2}\xi\Omega_{(AB}{}^{EL}\Omega_{CD)EL} + \tfrac{3}{2}\xi^{ELFH}\Omega_{E(ABC}\Omega_{D)LFH} \\
& - 2\Omega_{E(BC}{}^H\xi_A{}^{ELF}\Omega_{D)LFH} + \tfrac{1}{4}\xi^{ELFH}\Omega_{ABCD}\Omega_{ELFH} - \tfrac{1}{3}K\xi\Omega_{ABCD} \\
& + \tfrac{1}{2}\xi_{(AB}{}^{EL}\Omega_{CD)}{}^{FH}\Omega_{ELFH} + \tfrac{2}{5}\xi_{(CD}\nabla_{AB)}K + \tfrac{12}{5}\xi_{E(BCD}\nabla_{A)}{}^E K \\
& - 3\Omega_{E(BCD}\nabla_{A)}{}^E\xi - \tfrac{3}{2}\Omega_{(A}{}^{ELF}\nabla_{CD}\xi_{B)ELF} - \tfrac{3}{2}\Omega_{F(A}{}^{EL}\nabla_D{}^F\xi_{BC)EL} \\
& - \tfrac{9}{2}\Omega_{(AB}{}^{EL}\nabla_D{}^F\xi_{C)ELF} - \tfrac{9}{2}\nabla_{L(D}\nabla_C{}^E\xi_{AB)E}{}^L - \tfrac{3}{2}\nabla_{L(D}\nabla^{EL}\xi_{ABC)E} \\
& - 6K\nabla_{E(D}\xi_{ABC)}{}^E + 3\Omega_{L(AB}{}^E\nabla^{LF}\xi_{CD)EF} - 3\Omega_{(ABC}{}^E\nabla^{LF}\xi_{D)ELF} \\
& - 3\kappa^{EL}\nabla_{L(D}\Psi_{ABC)E} + 3\kappa_{(A}{}^E\nabla_D{}^L\Psi_{BC)EL}. \qquad (19c)
\end{aligned}
$$

What is required in the result?

CA ingredients:

- Split of the equations to obtain necessary conditions.
- Analysis of the interdependences of the equations to obtain a minimal set of necessary conditions.
- Analysis of the sufficiency of the conditions —this requires the construction of *propagation equations* for the Killing spinor equation.
- Ancillary: decomposition in terms of irreducible components.

What is required in the result?

CA ingredients:

- Split of the equations to obtain necessary conditions.
- Analysis of the interdependences of the equations to obtain a minimal set of necessary conditions.
- Analysis of the sufficiency of the conditions —this requires the construction of *propagation equations* for the Killing spinor equation.
- Ancillary: decomposition in terms of irreducible components.

Analytical ingredients:

- Propagation equations which are homogeneous in the Killings spinor equation, so that the result follows by uniqueness of the solutions to this type of equations.

The Killing spinor initial data equations

Theorem (T Bäckdahl & JAVK, also A García-Parrado & JAVK)

The development $(\mathcal{M}, g_{\mu\nu})$ *of an initial data set for the vacuum Einstein field equations,* $(\mathcal{S}, h_{ij}, K_{ij})$, *has a Killing spinor if and only if on* $\mathcal{S}$ *there exists a symmetric spinor* κ_{AB} *satisfying the equations:*

$$\nabla_{(AB}\kappa_{CD)} = 0, \qquad \textit{Spatial Killing Spinor Equation}$$

$$\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0, \qquad \textit{Algebraic Condition 1}$$

$$3\kappa_{(A}{}^{E}\nabla_{B}{}^{F}\Psi_{CD)EF} + \Psi_{(ABC}{}^{F}\xi_{D)F} = 0, \qquad \textit{Algebraic Condition 2}$$

where

$$\xi_{AB} \equiv \frac{3}{2}\nabla_{(A}{}^{Q}\kappa_{B)Q},$$

and ∇_{AB} *is the Sen connection.*

The Killing spinor initial data equations

Theorem (T Bäckdahl & JAVK, also A García-Parrado & JAVK)

The development $(\mathcal{M}, g_{\mu\nu})$ of an initial data set for the vacuum Einstein field equations, $(\mathcal{S}, h_{ij}, K_{ij})$, has a Killing spinor if and only if on $\mathcal{S}$ there exists a symmetric spinor κ_{AB} satisfying the equations:

$$\nabla_{(AB}\kappa_{CD)} = 0, \qquad \textit{Spatial Killing Spinor Equation}$$

$$\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0, \qquad \textit{Algebraic Condition 1}$$

$$3\kappa_{(A}{}^{E}\nabla_{B}{}^{F}\Psi_{CD)EF} + \Psi_{(ABC}{}^{F}\xi_{D)F} = 0, \qquad \textit{Algebraic Condition 2}$$

where

$$\xi_{AB} \equiv \frac{3}{2}\nabla_{(A}{}^{Q}\kappa_{B)Q},$$

and ∇_{AB} is the Sen connection.

Note:

The spinor $\Psi_{ABCD} = E_{ABCD} + \mathrm{i}B_{ABCD}$ can be calculated from initial data on $\mathcal{S}$ via

$$E_{ABCD} = -r_{(ABCD)} - \frac{1}{2}K_{AB}{}^{PQ}K_{CDPQ} + \frac{1}{2}K_{ABCD}K,$$

$$B_{ABCD} = -\mathrm{i}\, D^{Q}{}_{(A}K_{BCD)Q}.$$

Constructing an invariant?

Observation:

- The Killing spinor equations are overdetermined and have, in general, no solution.

Constructing an invariant?

Observation:

- The Killing spinor equations are overdetermined and have, in general, no solution.

Question:

- Can one *suitably generalise* the conditions so that they always admit a solution?
- Can one quantify the how much is missing for these equations to be satisfied (*defect* or *non-Kerness*)?

The spatial Killing spinor operator

A closer look:

- The operator in the equation

$$\nabla_{(AB}\kappa_{CD)} = 0,$$

  is elliptic overdetermined.
- Its formal adjoint of the above operator is given by

$$\nabla^{CD}\xi_{ABCD} - \Omega^{CDF}{}_{(A}\nabla_{B)F}\kappa_{CD}, \quad \Omega_{ABCD} \equiv K_{(ABCD)}.$$

  It is elliptic underdetermined.

The spatial Killing spinor operator

A closer look:

- The operator in the equation

$$\nabla_{(AB}\kappa_{CD)} = 0,$$

is elliptic overdetermined.

- Its formal adjoint of the above operator is given by

$$\nabla^{CD}\xi_{ABCD} - \Omega^{CDF}{}_{(A}\nabla_{B)F}\kappa_{CD}, \quad \Omega_{ABCD} \equiv K_{(ABCD)}.$$

It is elliptic underdetermined.

An elliptic equation!

- The composition of the above two operators renders the equation

$$L(\kappa_{AB}) \equiv \nabla^{CD}\nabla_{(AB}\kappa_{CD)} - \Omega^{CDF}{}_{(A}\nabla_{|DF|}\kappa_{B)C} - \Omega^{CDF}{}_{(A}\nabla_{B)F}\kappa_{CD} = 0,$$

which is elliptic. We call it the approximate spatial Killing spinor equation.

Asymptotically Euclidean data

Assumptions:

We assume $(\mathcal{S}, h_{ij}, K_{ij})$ to be smooth and with 2 asymptotically Euclidean ends. In every end it is assumed that there are asymptotically Cartesian coordinates such that:

$$h_{ij} = -\left(1 + \frac{2m}{r}\right)\delta_{ij} + o_\infty(r^{-3/2}),$$

$$K_{ij} = o_\infty(r^{-5/2}).$$

Notes:

- Standard Kerr data is included.

An existence result

An Ansatz for the solutions:

- Let
$$\kappa_{AB} = \mathring{\kappa}_{AB} + \theta_{AB}, \quad \theta_{AB} = o_\infty(r^{-1/2}),$$
where
$$\mathring{\kappa}_{AB} = \mp\frac{\sqrt{2}}{3}x_{AB} \mp \frac{2\sqrt{2}m}{3r}x_{AB}, \quad \text{at every end.}$$

An existence result

An Ansatz for the solutions:

- Let

$$\kappa_{AB} = \mathring{\kappa}_{AB} + \theta_{AB}, \quad \theta_{AB} = o_\infty(r^{-1/2}),$$

where

$$\mathring{\kappa}_{AB} = \mp \frac{\sqrt{2}}{3} x_{AB} \mp \frac{2\sqrt{2}m}{3r} x_{AB}, \quad \text{at every end.}$$

Theorem

There exists a smooth unique solution to

$$\nabla^{CD}\nabla_{(AB}\kappa_{CD)} - \Omega^{CDF}{}_{(A}\nabla_{|DF|}\kappa_{B)C} - \Omega^{CDF}{}_{(A}\nabla_{B)F}\kappa_{CD} = 0$$

with the asymptotic behaviour given by the Ansatz.

An existence result

An Ansatz for the solutions:

- Let

$$\kappa_{AB} = \mathring{\kappa}_{AB} + \theta_{AB}, \quad \theta_{AB} = o_\infty(r^{-1/2}),$$

where

$$\mathring{\kappa}_{AB} = \mp \frac{\sqrt{2}}{3} x_{AB} \mp \frac{2\sqrt{2}m}{3r} x_{AB}, \quad \text{at every end.}$$

Theorem

There exists a smooth unique solution to

$$\nabla^{CD}\nabla_{(AB}\kappa_{CD)} - \Omega^{CDF}{}_{(A}\nabla_{|DF|}\kappa_{B)C} - \Omega^{CDF}{}_{(A}\nabla_{B)F}\kappa_{CD} = 0$$

with the asymptotic behaviour given by the Ansatz.

Observation:

- Ancillary results to prove this result require heavy computer algebra computations —again carried out in `xAct`.

What is required in this result?

Analytic ingredients:

- Ellipticity of the equation.
- Fredholm alternative —analysis of the possible obstructions to existence of a solution.
  - Have to analyse the Kernel of the operator!

What is required in this result?

Analytic ingredients:

- Ellipticity of the equation.
- Fredholm alternative —analysis of the possible obstructions to existence of a solution.
  - Have to analyse the Kernel of the operator!

CA ingredients:

- The analysis of the asymptotic behaviour of the solution (which is used as Ansatz) requires CA computations.
- The analysis of the Kernel requires heavy computer algebra computations:
  - Have to show that the third covariant derivative of

$$\xi_{ABCD} \equiv \nabla_{(AB}\kappa_{CD)},$$

can be expressed in terms of lower order derivatives. From here it follows that the Kernel is trivial.

Constructing the invariant

Ingredients:

- Let κ_{AB} be a solution to the elliptic equation.
- In addition, let

$$J \equiv \int_{\mathcal{S}} \nabla_{(AB}\kappa_{CD)} \widehat{\nabla^{AB}\kappa^{CD}} \mathrm{d}\mu,$$

$$I_1 \equiv \int_{\mathcal{S}} |\text{Algebraic Condition 1}|^2 \mathrm{d}\mu,$$

$$I_2 \equiv \int_{\mathcal{S}} |\text{Algebraic Condition 2}|^2 \mathrm{d}\mu$$

Constructing the invariant

Ingredients:

- Let κ_{AB} be a solution to the elliptic equation.
- In addition, let

$$J \equiv \int_{\mathcal{S}} \nabla_{(AB}\kappa_{CD)} \widehat{\nabla^{AB}\kappa^{CD}} \mathrm{d}\mu,$$

$$I_1 \equiv \int_{\mathcal{S}} |\text{Algebraic Condition 1}|^2 \mathrm{d}\mu,$$

$$I_2 \equiv \int_{\mathcal{S}} |\text{Algebraic Condition 2}|^2 \mathrm{d}\mu$$

The invariant:

$$I \equiv J + I_1 + I_2.$$

The main result

Theorem

Given an asymptotically Euclidean initial data set for the Einstein vacuum field equations such that $\Psi_{ABCD} \neq 0$, $\Psi_{ABCD}\Psi^{ABCD} \neq 0$, *the invariant* I *vanishes if and only if the data set is data for the Kerr spacetime.*

The main result

Theorem

Given an asymptotically Euclidean initial data set for the Einstein vacuum field equations such that $\Psi_{ABCD} \neq 0$, $\Psi_{ABCD}\Psi^{ABCD} \neq 0$, *the invariant* I *vanishes if and only if the data set is data for the Kerr spacetime.*

Observations:

- The invariant I measures the non-Kerrness of the slice $(\mathcal{S}, h_{ij}, K_{ij})$.
- The construction can be extended to other settings —e.g. hyperboloids, outer domains of communication.

The main result

Theorem

Given an asymptotically Euclidean initial data set for the Einstein vacuum field equations such that $\Psi_{ABCD} \neq 0$, $\Psi_{ABCD}\Psi^{ABCD} \neq 0$, the invariant I vanishes if and only if the data set is data for the Kerr spacetime.

Observations:

- The invariant I measures the non-Kerrness of the slice $(\mathcal{S}, h_{ij}, K_{ij})$.
- The construction can be extended to other settings —e.g. hyperboloids, outer domains of communication.

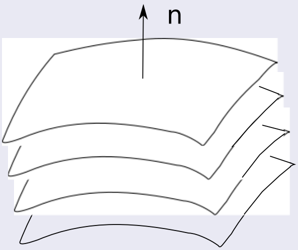
Contact with numerical Relativity:

- The construction of the invariant is fully amenable to a numerical implementation.
- Requires solving a system of 6 coupled linear elliptic PDE's.
- The use of spinor is non-essential. Everything can be done tensorially.

Evolution of the invariant?

A sequence of non-Kerrness:

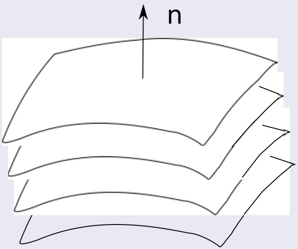
- Consider a foliation of spacetime.
- On each leave evaluate the invariant:
  - One obtains a sequence of non-negative numbers I_t.
- What can be said about the sequence?
  - Does it satisfy some monotonic behaviour?
  - Under what conditions?
  - Is the sequence foliation-dependent?



Evolution of the invariant?

A sequence of non-Kerrness:

- Consider a foliation of spacetime.
- On each leave evaluate the invariant:
    - One obtains a sequence of non-negative numbers I_t.
- What can be said about the sequence?
    - Does it satisfy some monotonic behaviour?
    - Under what conditions?
    - Is the sequence foliation-dependent?



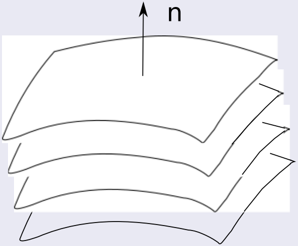
Input from Numerics:

- These issues would benefit from insight from numerical simulations!

Evolution of the invariant?

A sequence of non-Kerrness:

- Consider a foliation of spacetime.
- On each leave evaluate the invariant:
  - One obtains a sequence of non-negative numbers I_t.
- What can be said about the sequence?
  - Does it satisfy some monotonic behaviour?
  - Under what conditions?
  - Is the sequence foliation-dependent?



Input from Numerics:

- These issues would benefit from insight from numerical simulations!

Feedback into mathematical GR:

- Can numerical simulations benefit from the information that I_t provides? What type of physical information can be extracted?

Concluding reflexions:

Global versus local

- Are global objects adequate for numerical investigations?
- Or rather, should one use something more local?

Concluding reflexions:

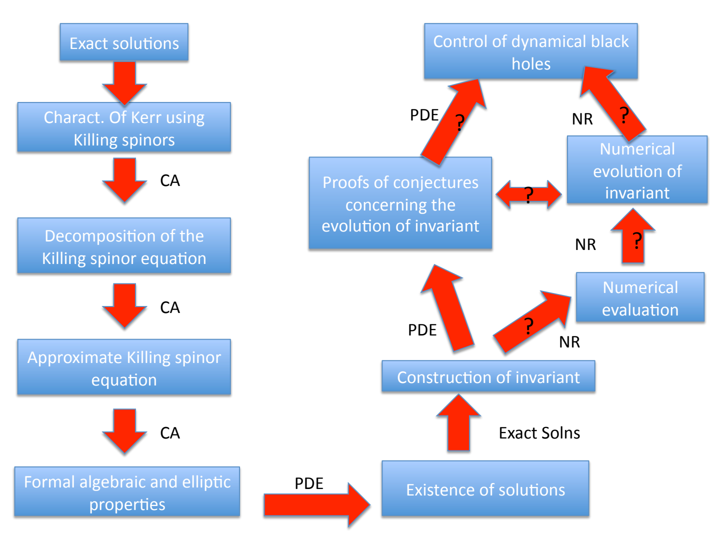
Global versus local

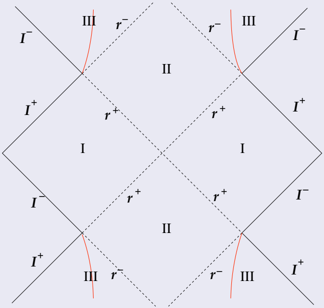
- Are global objects adequate for numerical investigations?
- Or rather, should one use something more local?

Which is the closest Kerr?

- The invariant discussed here quantifies non-Kerness... However, given $(\mathcal{S}, h_{ij}, K_{ij})$ can one say which is the closest (or optimal) Kerr solution?
  - This is of relevance for perturbative methods!

The programme in a snapshot:





References:

- A García-Parrado & JA Valiente Kroon. *Killing spinor initial data.* J. Geom. Phys. **58**, 1186 (2008). Also at `arXiv:0805.4505[gr-qc]`
- T Bäckdahl & JA Valiente Kroon. *Geometric invariant measuring the deviation from Kerr data.* Phys. Rev. Lett **104**, 231102 (2010). Also at `arXiv:1001.4492[gr-qc]`
- T Bäckdahl & JA Valiente Kroon. *On the construction of a geometric invariant measuring the deviation from Kerr data.* Available at `arXiv:1005.0743[gr-qc]`