(Conformally) semisymmetric spaces and special semisymmetric Weyl tensors

S Brian Edgar

Abstract

Semisymmetric spaces are a natural generalisation of symmetric spaces. For semisymmetric spaces in four dimensions with Lorentz signature, the Weyl tensor is easily seen (via spinors) to have a particularly simple quadratic property, which we call a special semisymmetric Weyl tensor. Using dimensionally dependent tensor identities, all (conformally) semisymmetric spaces are confirmed to have special semisymmetric Weyl tensors for all signatures in four dimensions. Furthermore, all Ricci-semisymmetric spaces with special semisymmetric Weyl tensors are shown to be semisymmetric for all signatures in four dimensions.

Counterexamples demonstrate that these two properties have no direct generalisations in higher dimensions.

Background

It is known, for spaces in dimensions:

1. $n \geq 5$ for all signatures
2. $n = 4$ for Lorentz signature (using spinors)

that *semi-symmetry* is equivalent to *conformal semi-symmetry* when the Weyl conformal tensor C is non-zero:

$$\nabla_{[a}\nabla_{b]}R_{cdef} = 0 \Longleftrightarrow \nabla_{[a}\nabla_{b]}C_{cdef} = 0 \quad (\text{if } C \neq 0)$$

For dimensions $n = 4$, a single simple counterexample is known, which has proper Riemannian signature. Using four-dimensional tensor identities, it is possible to show precisely when the result fails for:

- proper Riemannian signature,
- neutral signature.

Spinor version $(n = 4)$

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semi-symmetric conditions:

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$$0 = \Box_{A'B'}\Psi_{CDEF} = \Phi_{A'B'(C}{}^{G}\Psi_{DEF)G} \tag{2}$$

$$0 = \Box_{AB}\Phi_{CDC'D'} = 2\mathrm{X}_{AB(C}{}^{E}\Phi_{D)EC'D'} + 2\Phi_{AB(C'}{}^{E'}\Phi_{|CD|D')E'} \tag{3}$$

The first two correspond to $\nabla_{[a}\nabla_{b]}C_{cdef} = 0$ while the third corresponds to $\nabla_{[a}\nabla_{b]}R_{cd} = 0$ ("Ricci semi-symmetry").

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(Eriksson & Senovilla CQG **27** 027001 (2010))

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The special cases —having a zero Weyl tensor — have been discussed by J Åman (arXiv:1006.5684)

‘Special semisymmetric’ Weyl tensor

Thus, one can concentrate on the expression (1)

$$0 = \Box_{AB}\Psi_{CDEF} = \mathrm{X}_{AB(C}{}^{G}\Psi_{DEF)G}$$

which is equivalent to

$$24\Psi_{AB(C}{}^{G}\Psi_{DEF)G} = -R\left(\varepsilon_{A(C}\Psi_{DEF)B} + \varepsilon_{B(C}\Psi_{DEF)A}\right).$$

Tensor expression:

$$C^{ef}{}_{k[a}C^{k}{}_{b]cd} + C^{ef}{}_{k[c}C^{k}{}_{d]ab} = \frac{R}{6}\left(\delta^{[e}_{[b}C_{a]}{}^{f]}{}_{cd} + \delta^{[e}_{[d}C_{c]}{}^{f]}{}_{ab}\right)$$

This can be straightforwardly generalised to all dimensions $n \geq 4$.

‘Special semisymmetric’ Weyl tensor

Definition (Edgar, 2010)

A Weyl tensor is said to be special semisymmetric if it satisfies:

$$C^{ef}{}_{k[a}C^{k}{}_{b]cd} + C^{ef}{}_{k[c}C^{k}{}_{d]ab} = \frac{2R}{n(n-1)}\left(\delta^{[e}_{[b}C_{a]}{}^{f]}{}_{cd} + \delta^{[e}_{[d}C_{c]}{}^{f]}{}_{ab}\right)$$

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Theorem (Edgar, 2010)

(Conformal) semi-symmetric spaces have special semi-symmetric Weyl tensors in —and only in— $n = 4$ dimensions, for all signatures.

Proof

Proof. The conformal semisymmetric condition reads:

$$C^{ef}{}_{k[a}C^{k}{}_{b]cd} + C^{ef}{}_{k[c}C^{k}{}_{d]ab} - \frac{2}{n(n-1)}R\left(\delta^{[e}_{[b}C_{a]}{}^{f]}{}_{cd} + \delta^{[e}_{[d}C_{c]}{}^{f]}{}_{ab}\right)$$

$$= \frac{2}{n-2}\left(\tilde{R}^{[e}_{i}\delta^{f]}_{[a}C^{i}{}_{b]cd} + \tilde{R}^{[e}_{i}\delta^{f]}_{[c}C^{i}{}_{d]ab} - \tilde{R}^{[e}_{[a}C^{f]}{}_{b]cd} - \tilde{R}^{[e}_{[c}C^{f]}{}_{d]ab}\right)$$

where $\tilde{R}_{ab} \equiv R_{ab} - \frac{R}{n}g_{ab}$ is the trace-free Ricci tensor.

This relation has a non-trivial trace:

$$0 = R^{ej}{}_{i[a}C^{i}{}_{b]cj} + R^{ej}{}_{i[c}C^{i}{}_{j]ab}$$

Use fddis (Edgar & Höglund, JMP 43 659 (2002))

This trace can be split into two parts:

$$0 = (n-1) C_{abi}{}^{(c} \tilde{R}^{d)i} - 2 C_i{}^{(cd)}{}_{[a} \tilde{R}_{b]}{}^{i} - 2\delta^{(c}_{[a} C^{d)i}{}_{b]}{}^{j} \tilde{R}_{ij}$$

$$C_{abij} C^{cdij} + 4 C_{[a}{}^{ij[c} C_{b]ij}{}^{d]} - \frac{2R}{n} C_{ab}{}^{cd} =$$
$$\frac{1}{2(n-2)} \left((n-3)\tilde{R}^{i[c} C^{d]}{}_{iab} + \tilde{R}^{i}{}_{[a} C_{b]i}{}^{cd} + 2\tilde{R}_{ij}\delta^{[c}_{[a} C^{d]i}{}_{b]}{}^{j} \right)$$

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But a basic fddi in $n=4$ is $C_{[ab}{}^{[cd}\delta^{j]}_{i]} = 0$, from where

$$0 = 9\tilde{R}^i{}_jC_{[ab}{}^{[cd}\delta^{j]}_{i]} =$$
$$2\left(\tilde{R}^{i[c}C^{d]}{}_{iab} + \tilde{R}^i{}_{[a}C_{b]i}{}^{cd} + 2\tilde{R}_{ij}\delta^{[c}_{[a}C^{d]i}{}_{b]}{}^j\right)$$

End of proof: Use more fddis!

Thus, the 2nd part of the trace becomes $(n = 4)$

$$C_{abij}C^{cdij} + 4C_{[a}{}^{ij[c}C_{b]ij}{}^{d]} - \frac{R}{2}C_{ab}{}^{cd} = 0\,. \quad (\star)$$

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Surprisingy, this implies by itself that the Weyl tensor is (in $n = 4$) special semi-symmetric: use again $C_{[ab}{}^{[cd}\delta^{j]}_{i]} = 0$ to build

$$C_i{}^j{}_{ef}C_{[ab}{}^{[cd}\delta^{j]}_{i]} + C_i{}^j{}_{ab}C_{[ef}{}^{[cd}\delta^{j]}_{i]} = 0$$

Managing this (and its trace) in a judicious manner, and using $(\star)$, one can arrive at ($n = 4$)

$$C^{ef}{}_{k[a}C^k{}_{b]cd} + C^{ef}{}_{k[c}C^k{}_{d]ab} = \frac{R}{6}\left(\delta^{[e}_{[b}C_{a]}{}^{f]}{}_{cd} + \delta^{[e}_{[d}C_{c]}{}^{f]}{}_{ab}\right)$$

This is the condition for the Weyl tensor to be special semi-symmetric.

Final Remarks

- There are explicit counterexamples showing that the above properties are exclusive of $n = 4$.

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- All the previous results hold for the more general classes of (conformally) *pseudo-symmetric* spaces, defined by

$$R^{ef}{}_{k[a}R^{k}{}_{b]cd} + R^{ef}{}_{k[c}R^{k}{}_{d]ab} = L\left(\delta^{[e}_{[b}R_{a]}{}^{f]}{}_{cd} + \delta^{[e}_{[d}R_{c]}{}^{f]}{}_{ab}\right)$$

$$R^{ef}{}_{k[a}C^{k}{}_{b]cd} + R^{ef}{}_{k[c}C^{k}{}_{d]ab} = \tilde{L}\left(\delta^{[e}_{[b}C_{a]}{}^{f]}{}_{cd} + \delta^{[e}_{[d}C_{c]}{}^{f]}{}_{ab}\right)$$

  for some scalars L and $\tilde{L}$.

Observe that they correspond respectively to

$$\nabla^{[e}\nabla^{f]}R_{abcd} = -L\left(\delta^{[e}_{[b}R_{a]}{}^{f]}{}_{cd} + \delta^{[e}_{[d}R_{c]}{}^{f]}{}_{ab}\right)$$

$$\nabla^{[e}\nabla^{f]}C_{abcd} = -\tilde{L}\left(\delta^{[e}_{[b}C_{a]}{}^{f]}{}_{cd} + \delta^{[e}_{[d}C_{c]}{}^{f]}{}_{ab}\right)$$

Thank you, Brian