Singularity theorems assuming trapped submanifolds of arbitrary dimension

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In Memoriam S. Brian Edgar (1945-2010)

Outline

1. Classical singularity theorems
2. The mean curvature vector and trapped submanifolds
3. Existence of points focal to submanifolds of arbitrary dimension
4. Main results: singularity theorems
5. Discussion with some applications

References

G J Galloway and JMMS, ***Singularity theorems based on trapped submanifolds of arbitrary co-dimension***, Class. Quantum Grav. **27** 152002 (2010) [arXiv:1005.1249]

The classical Hawking-Penrose singularity theorem

Theorem (Hawking and Penrose)

Spacetime (<u>in 4 dimensions</u>) is causal geodesically incomplete if the strong-energy, causality and generic conditions hold and if there is one of the following:

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What about co-dimension 3 —a closed spacelike curve?

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$\Longrightarrow$ The mean curvature vector $\vec{H}$!

The mean curvature vector: trapped submanifolds

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$$\Phi : \zeta \longrightarrow \mathcal{V} \qquad x^\mu = \Phi^\mu(\lambda^A). \quad A, B, \dots = m+1, \dots, n$$

Thus, the tangent vectors (seen on $\mathcal{V}$) are:

$$\vec{e}_A \equiv \Phi'(\partial_{\lambda^A}) \Longleftrightarrow e^\mu_A = \frac{\partial \Phi^\mu}{\partial \lambda^A}$$

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First fundamental form:

$$\gamma_{AB}(\lambda) = g|_\zeta(\vec{e}_A, \vec{e}_B) = g_{\mu\nu}(\Phi) e^\mu_A e^\nu_B$$

is *positive-definite*. So, ζ is assumed to be SPACELIKE.

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Then, $\forall x \in \zeta$

$$T_x\mathcal{V} = T_x\zeta \oplus T_x\zeta^\perp$$

called *tangent* and *normal* parts.

Notation: extrinsic curvature

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The second fundamental form of ζ in $(\mathcal{V}, g)$

relative to any $\vec{n} \in \mathfrak{X}(\zeta)^{\perp}$ is:

$$K_{AB}(\vec{n}) \equiv n_\mu K^{\mu}_{AB} \, .$$

These are 2-covariant symmetric tensor fields on ζ.

At each point on ζ there are m linearly independent normal one-forms. If $m > 1$ all of these can be chosen to be null if desired.

Mean curvature vector. Expansions

The *mean curvature vector:*

$$\mathfrak{X}(S)^{\perp} \ni \vec{H} \equiv \gamma^{AB}\vec{K}_{AB}$$

The expansion of ζ in $(\mathcal{V}, g)$

relative to any $\vec{n} \in \mathfrak{X}(\zeta)^{\perp}$ is:

$$\theta(\vec{n}) \equiv n_{\mu}H^{\mu} = \gamma^{AB}K_{AB}(\vec{n}).$$

Future-trapped subamanifolds: $\vec{H}$ is future on ζ

Definition (Trapped submanifold)

A spacelike submanifold ζ is said to be future trapped *(f-trapped from now on) if $\vec{H}$ is timelike and future-pointing everywhere on ζ, and similarly for past trapped.*

Equivalently

$\theta(\vec{n}) < 0$ for <u>every</u> future pointing normal $\vec{n}$.

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$\vec{H}$	Type of surface
zero	stationary or minimal
null and future	marginally f-trapped
causal and future	weakly f-trapped
timelike future	f-trapped

Notation

- n_μ: *future-pointing* normal to the spacelike submanifold ζ,
- γ: geodesic curve tangent to n^μ at ζ
- u: affine parameter along γ ($u = 0$ at ζ).
- N^μ: geodesic vector field tangent to γ ($N_\mu|_{u=0} = n_\mu$).
- $\vec{E}_A$: vector fields defined by parallelly propagating $\vec{e}_A$ along γ ($\vec{E}_A|_{u=0} = \vec{e}_A$)
- $P^{\nu\sigma} \equiv \gamma^{AB} E^\nu_A E^\sigma_B$ (at $u = 0$ this is the projector to ζ).

Existence of focal points

Proposition

Let ζ be a spacelike submanifold of co-dimension m, and let n_μ be a future-pointing normal to ζ. If $\theta(\vec{n}) \equiv (m-n)c < 0$ and the curvature tensor satisfies the inequality

$$\boxed{R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} \geq 0} \quad (1)$$

along γ, then there is a point focal to ζ along γ at or before $\gamma|_{u=1/c}$, provided γ is defined up to that point.

Remarks:

1. Spacelike hypersurfaces: $m = 1$, there is a unique timelike orthogonal direction n_μ. Then $P_{\mu\nu} = g_{\mu\nu} - (N_\rho N^\rho)^{-1} N_\mu N_\nu$ and (1) reduces to

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2. Spacelike 'surfaces': $m = 2$, there are two independent null normals on ζ, say n_μ and ℓ_μ. (Define L_μ parallelly propagating ℓ_μ on γ). Then, $P_{\mu\nu} = g_{\mu\nu} - (N_\rho L^\rho)^{-1}(N_\mu L_\nu + N_\nu L_\mu)$ and again (1) reduces to

$$R_{\mu\nu} N^\mu N^\nu \geq 0$$

(the *null convergence condition* along γ).

The curvature condition

For co-dimension $m > 2$, the interpretation of condition (1) can be given physically in terms of tidal forces, or geometrically in terms of sectional curvatures.

Timelike *unit* normal n_μ

Sectional curvature $k(n,e)$ relative to the plane $\langle \vec{n}, \vec{e} \rangle$ $(n_\mu e^\mu = 0)$

$$R_{\mu\nu\rho\sigma} n^\mu e^\nu n^\rho e^\sigma = k(n,e)(n_\rho n^\rho)(e_\rho e^\rho) = -k(n,e)(e_\rho e^\rho)$$

Hence (1): *the sum of the $n - m$ sectional curvatures relative to a set of independent and mutually orthogonal timelike planes aligned with n_μ is non-positive, and remains so along γ.*

In physical terms, this is a statement about the attractiveness of the gravitational field on average. **The tidal force in directions initially tangent to ζ is attractive on average**.

The curvature condition

Null normal n_μ

For a null normal n_μ one may consider analogously,

$$R_{\mu\nu\rho\sigma} n^\mu e^\nu n^\rho e^\sigma = -k(n, e)(e_\rho e^\rho)$$

where $n_\mu e^\mu = 0$, and $k(n, e)$ is called the *null* sectional curvature relative to the plane spanned by $\vec{n}$ and $\vec{e}$.

Hence (1): *the sum of the $n - m$ null sectional curvatures relative to a set of independent and mutually orthogonal null planes aligned with n_μ is non-positive, and remains so along γ.*

The Penrose singularity theorem

Recall: $E^+(\zeta) \equiv J^+(\zeta) \backslash I^+(\zeta)$

Proposition (Intermediate result)

Let ζ be a closed f-trapped submanifold of co-dimension $m > 1$, and assume that

$$R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} \geq 0$$

for any future-pointing null normal n^μ. Then, either $E^+(\zeta)$ is compact, or the spacetime is future null geodesically incomplete, or both.

Remark: The case with $m = 1$ is not included here because it is trivial. If ζ is a spacelike hypersurface, then $E^+(\zeta) \subset \zeta$ —and actually $E^+(\zeta) = \zeta$ if ζ is achronal—, and the compactness of $E^+(\zeta)$ follows readily without any further assumptions.

The Penrose singularity theorem

Theorem (Generalized Penrose singularity theorem)

If $(\mathcal{V}, g)$ *contains a non-compact Cauchy hypersurface* Σ *and a closed f-trapped submanifold* ζ *of arbitrary co-dimension, and if*

$$R_{\mu\nu\rho\sigma}N^{\mu}N^{\rho}P^{\nu\sigma} \geq 0$$

holds along every future-directed null geodesic emanating orthogonally from ζ*, then* $(\mathcal{V}, g)$ *is future null geodesically incomplete.*

The Hawking-Penrose singularity theorem

Proposition (Intermediate result)

If $(\mathcal{V}, g)$ is strongly causal and there is a closed f-trapped submanifold ζ of arbitrary co-dimension $m > 1$ such that

$$R_{\mu\nu\rho\sigma}N^{\mu}N^{\rho}P^{\nu\sigma} \geq 0$$

holds along every null geodesic emanating orthogonally from ζ, then either $E^{+}(E^{+}(\zeta) \cap \zeta)$ is compact, or the spacetime is null geodesically incomplete, or both.

The Hawking-Penrose singularity theorem

Theorem (Generalized Hawking-Penrose singularity theorem)

If the chronology, generic and strong energy conditions hold and there is a closed f-trapped submanifold ζ *of arbitrary co-dimension such that*

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along every null geodesic emanating orthogonally from ζ *then the spacetime is causal geodesically incomplete.*

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- Spacelike 'surfaces' $m = 2$: Condition (1) is actually included in the strong energy condition.
- Points $m = n$: The 'same' happens.

These three cases cover the original Hawking-Penrose theorem.

Selected applications

- The main application of these theorems is, of course, to higher dimensional spacetimes
- thus, they are usable in Kaluza-Klein, string, supergravity, M-type ... theories.
- In dimension 11, say, there are now 10 different possibilities for the boundary condition in the theorems
- this can have relevance in connection with the *compactified* extra-dimensions.

Example: instability of spatial extra dimensions?

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- Those problems can be avoided by using the generalized Theorems. It is enough that the compact extra-dimensional space, or *any* of its compact less-dimensional subsets, satisfy the trapping condition
- Hence, the basic argument of Penrose acquires a wider applicability and requires less restrictions.

Example in 4D: cylindrical symmetry

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- An obvious relevant example is the case of spacetimes with whole cylindrical symmetry

  $$ds^2 = -A^2dt^2 + B^2d\rho^2 + F^2d\varphi^2 + E^2dz^2,$$

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  where $\partial_\varphi, \partial_z$ are spacelike commuting Killing vectors. The coordinate φ is closed with standard periodicity 2π.
- The cylinders with constant t and ρ are geometrically preferred; however, they are *not* compact in general

Example in 4D: cylindrical symmetry

- Nevertheless, the spacelike curves with constant values of t, ρ and z are certainly *closed*. Their mean curvature vector is proportional to dF. Thus, the causal character of the gradient of $g(\partial_\varphi, \partial_\varphi)$ characterizes the trapping of these closed circles.

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- Thereby, many results on incompleteness of geodesics can be found.
- Moreover, there arises a new hypersurface, defined as the set of points where dF is null, which is a new type of horizon, being a boundary separating the trapped from the untrapped circles, and containing marginally trapped circles.

Appplication: asymptotically de Sitter cosmologies

Theorem

Let $(\mathcal{V}, g)$ *have all null sectional curvatures non-negative. Suppose* Σ *is a compact Cauchy hypersurface for* $(\mathcal{V}, g)$ *which is expanding to the future in all directions, i.e., which has positive definite second fundamental form with respect to the future pointing normal. Then, if* $\pi_1(\Sigma)$ *has non-finite cardinality,* $(\mathcal{V}, g)$ *is past null geodesically incomplete.*

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Remarks:

- The timelike convergence condition is not assumed.
- Observe that the timelike convergence condition does not in general hold in spacetimes which satisfy the Einstein equations with positive cosmological constant
- On the other hand, our condition (1) on tidal forces is satisfied strictly in the FLRW models, as well as in sufficiently small perturbations of those models.

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- Since Σ has negative definite second fundamental form with respect to the *past* pointing unit normal, one easily verifies that σ is a past-trapped circle in $(\mathcal{V}, g)$.

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- Since Σ has negative definite second fundamental form with respect to the *past* pointing unit normal, one easily verifies that σ is a past-trapped circle in $(\mathcal{V}, g)$.
- Then, the Penrose theorem can be applied. Since all the Cauchy hypersurfaces of $(\mathcal{V}, g)$ are compact this does not directly lead to geodesic incompleteness.
- However, passing to a covering spacetime one can get the result.