# Hyperbolicity of Hamiltonian formulations in General Relativity

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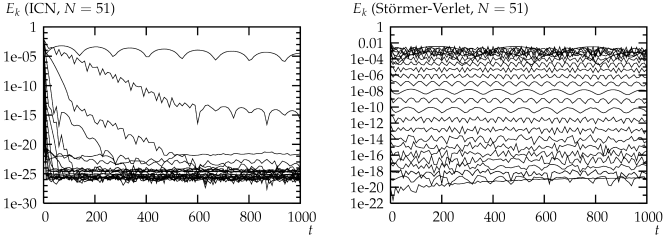
# Symplectic integrators

- In previous studies we found that symplectic integration can be favorable for numerical GR:




- We want to apply these methods in the general case
  $\longrightarrow$ need a formulation with Hamilton structure.
- Currently known: ADM, generalized harmonic (GHG) and others

# Well-Posedness and Hyperbolicity

- Problems with existing formulations:
  - Dynamical subsystem does not possess a well-posed initial (boundary) value problem (I(B)VP) (ADM)
  - Nothing is known about well-posedness of the IBVP
  - Not singularity avoiding $\rightarrow$ need to use excision (GHG)
- Can we find a formulation with a well-posed I(B)VP that allows the puncture gauge and has Hamiltonian structure?
- A system has a well-posed I(B)VP if it is strongly (symmetric) hyperbolic (with appropriate boundary conditions)
- We find a symmetric hyperbolic formulation with small modifications of the puncture gauge

# Modified Hamiltonian and equations of motion

- Without Hamilton structure one can choose gauge conditions and add constraints to the equations of motion arbitrarily (without effect on the physics)
- Due to Hamilton structure the two aspects are coupled
- Get new system by adding gauge terms to the ADM Hamiltonian $\rightarrow$ equations of motion
- We consider formulations where the shift appears in advection and lower order terms only ($\partial_t u = \beta^i \partial_i u + \ldots$)
  $\rightarrow$ seven different gauge terms that change the principal part
  $\rightarrow$ seven parameters $C_1, \ldots, C_7$

# Linear Algebra

- To obtain formulations with well-posed I(B)VP we ask for their hyperbolicity
- This can be reduced to the analysis of particular matrices
    - strong hyperbolicity: principal symbol
    - symmetric hyperbolicity: principal part matrix
- The structure of the equations is such that analysis can be reduced to $2 \times 2$ blocks

# Results

- We find $3+1$ families of strongly hyperbolic formulations
- There are strongly hyperbolic formulations which are not symmetric hyperbolic
- We find several 2-parameter families of symmetric hyperbolic formulations
- For the unifying 3-parameter family expressions become too complicated to prove symmetric hyperbolicity

# Puncture gauge

- One can choose the parameters as scalar functions of metric, lapse and shift without changing the principal part
- One can get symmetric hyperbolic formulations close to puncture gauge. In Z4 variables:

$$\partial_t \alpha = \beta^i \partial_i \alpha - \mu_L \alpha^2 K + \frac{1}{2}\mu_L \alpha^2 \Theta,$$

$$\partial_t \beta^i = \beta^j \partial_j \beta^i + \mu_S \gamma^{1/3} \Gamma^i_{jk}\gamma^{jk} + \frac{1}{3}(\mu_S \gamma^{1/3} - \alpha^2)\Gamma^k_{kj}\gamma^{ij} + 2\mu_S \gamma^{1/3} Z^i - \alpha D^i \alpha, \tag{1}$$

+evolution equations for $\gamma_{ij}, K_{ij}, \Theta$ and $Z_i$

- Compare with puncture gauge:

$$\partial_t \alpha = \beta^i \partial_i \alpha - \mu_L \alpha^2 K, \tag{2}$$

$$\partial_t \beta^i = \beta^j \partial_j \beta^i + \mu_S \gamma^{1/3} \Gamma^i_{jk}\gamma^{jk} + \frac{1}{3}\mu_S \gamma^{1/3}\Gamma^k_{kj}\gamma^{ij} - \eta\beta^i.$$

# Summary

- We are interested in the well-posedness of Hamiltonian formulations of GR
- We analyze a large class of formulations according to their hyperbolicity level and identify all strongly hyperbolic and some symmetric hyperbolic formulations
- With appropriate choice of parameters we come close the the puncture gauge conditions

# Future work and Further reading

- Techniques for analysis of strong hyperbolicity can be applied to non Hamiltonian systems in the same variables as well
$\rightarrow$ 15 parameters, up to $142 + x$ families of strongly hyperbolic formulations
- Investigate numerical implementation of the symmetric hyperbolic puncture system (with symplectic and non symplectic methods)
$\rightarrow$ at the moment we find divergencies near the puncture

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*Hyperbolic formulations of General Relativity with Hamiltonian structure*
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