$f(G)$ modified gravity and the energy conditions

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Introduction

- Cosmology is thriving in a golden age, where a central theme is the perplexing fact that the Universe is undergoing an accelerating expansion.
- This represents a new imbalance in the governing gravitational equations.
- Historically, physics has addressed such imbalances by either identifying sources that were previously unaccounted for, or by altering the governing equations.
- The standard model of cosmology has favored the first route to addressing the imbalance by a missing stress-energy component.
- One may also explore the alternative viewpoint, namely, through a modified gravity approach.

Motivations

- Thus, a promising way to explain these major problems is to assume that at large scales GR breaks down, and a more general action describes the gravitational field.
- Generalizations of the Einstein-Hilbert Lagrangian have been proposed, involving second order curvature invariants such as R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, $\varepsilon^{\alpha\beta\mu\nu}R_{\alpha\beta\gamma\delta}R^{\gamma\delta}_{\mu\nu}$, $C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}$, etc.
- Physical motivations for these modifications of gravity: possibility of a more realistic representation of the gravitational fields near curvature singularities and to create some first order approximation for the quantum theory of gravitational fields.

Motivations

1. In addition to this, considering higher-order gravity, the motivations should be consistent with several quantum gravity candidates.
2. String/M-theory predict unusual gravity-matter couplings.
3. Couple a scalar field with higher order invariants.
4. String/M-theory predict scalar field couplings with the Gauss-Bonnet invariant important in the appearance of non-singular early time cosmologies.
5. Apply these motivations to the late-time Universe!

Gauss-Bonnet gravity

- Action of Gauss-Bonnet gravity:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa} - \frac{\lambda}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + f(\phi)\mathcal{G}\right] + S_M(g^{\mu\nu},\psi)\,, \tag{1}$$

- $\lambda = +1$ is defined for a canonical scalar field, $\lambda = -1$ for a phantom field, respectively.
- The Gauss-Bonnet invariant: $\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$.
- It is possible to construct models that account for the late-time cosmic acceleration.

Modified Gauss-Bonnet gravity

- Action of modified Gauss-Bonnet gravity:

$$S=\int d^4x\sqrt{-g}\left[\frac{R}{2\kappa}+f(\mathcal{G})\right]+S_M(g^{\mu\nu},\psi)\,. \qquad (2)$$

- Introduce two auxiliary scalar fields A and B:

$$S=\int d^4x\sqrt{-g}\left[\frac{R}{2\kappa}+B(\mathcal{G}-A)+f(A)\right]\,. \qquad (3)$$

- Varying with respect to B, one obtains $A=\mathcal{G}$.
  Varying with respect to A, one obtains $B=f'(A)$.
- Define: $\phi=A\,,\mathrm{and}\,V(\phi)=Af'(A)-f(A)$,
- One finally ends up with the following action

$$S=\int d^4x\sqrt{-g}\left[\frac{R}{2\kappa}-V(\phi)+f(\phi)\mathcal{G}\right]+S_M(g^{\mu\nu},\psi)\,. \qquad (4)$$

(action for Gauss-Bonnet gravity without the kinetic term).

Field equations

- In the flat FRW background, the field equations for $f(G)$-gravity are given by

$$24H^3\dot{f}'(G) + 6H^2 + f(G) - Gf'(G) = 2\kappa^2\rho\,, \tag{5}$$

$$8H^2\ddot{f}'(G) + 16H\dot{f}'(G)\left(\dot{H} + H^2\right) + \left(4\dot{H} + 6H^2\right) + f(G) - Gf'(G) = -2\kappa^2 p\,, \tag{6}$$

(overdot $\equiv$ derivative with respect t).

- Definitions: $R = 6(2H^2 + \dot{H})$ and $G = 24H^2(H^2 + \dot{H})$.
- Effective gravitational field equations:

$$\rho_{\rm eff} = \frac{3}{\kappa^2}H^2\,, \quad p_{\rm eff} = -\frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right)\,,$$

where $\rho_{\rm eff}$ and $p_{\rm eff}$ are the effective energy density and pressure, respectively, defined as

- $$\rho_{\rm eff} = \rho + \frac{1}{2\kappa^2}\Big[-f(G) + 24H^2\left(H^2+\dot{H}\right)f'(G) -24^2H^4\left(2\dot{H}^2 + H\ddot{H} + 4H^2\dot{H}\right)f''(G)\Big]\,, \quad (7)$$

- $$p_{\rm eff} = p + \frac{1}{2\kappa^2}\Big\{f(G) - 24H^2\left(H^2+\dot{H}\right)f'(G) +(24)8H^2\Big[6\dot{H}^3 + 8H\dot{H}\ddot{H} + 24\dot{H}^2H^2 +6H^3\ddot{H} + 8H^4\dot{H} + H^2\dddot{H}\Big]f''(G) +8(24)^2H^4\left(2\dot{H}^2 + H\ddot{H} + 4H^2\dot{H}\right)^2 f'''(G)\Big\}\,. \quad (8)$$

Raychaudhuri equation

- The energy conditions arise when one refers to the Raychaudhuri equation for the expansion.
- Note that the Raychaudhuri equation is a purely geometric statement, and as such it makes no reference to any gravitational field equations.
- The condition for attractive gravity reduces to $R_{\mu\nu}k^\mu k^\nu \geq 0$
- However, in general relativity, through the Einstein field equation one can write the above condition in terms of the stress-energy tensor given by $T_{\mu\nu}k^\mu k^\nu \geq 0$.
- However, in $f(G)$ modified theories of gravity the field equation is written as the following effective gravitational field equation

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = T^{\mathrm{eff}}_{\mu\nu}\,, \tag{9}$$

Effective field equations

- Thus, the positivity condition, $R_{\mu\nu}k^\mu k^\nu \geq 0$, provides the following form for the null energy condition $T^{\rm eff}_{\mu\nu}k^\mu k^\nu \geq 0$, through the modified gravitational field equation.
- As the Raychaudhuri equation holds for any geometrical theory of gravitation, we will maintain its physical motivation, namely, the focussing of geodesic congruences, along with the attractive character of gravity to deduce the energy conditions in the context of $f(G)$ modified gravity.

Effective field equations

- Thus, using the modified (effective) gravitational field equations the NEC, WEC, SEC and the DEC are given by

$$\mathrm{NEC} \Longleftrightarrow \rho_{\mathrm{eff}} + p_{\mathrm{eff}} \geq 0 \tag{10}$$

$$\mathrm{WEC} \Longleftrightarrow \rho_{\mathrm{eff}} \geq 0 \text{ and } \rho_{\mathrm{eff}} + p_{\mathrm{eff}} \geq 0 \tag{11}$$

$$\mathrm{SEC} \Longleftrightarrow \rho_{\mathrm{eff}} + 3p_{\mathrm{eff}} \geq 0 \text{ and } \rho_{\mathrm{eff}} + p_{\mathrm{eff}} \geq 0 \tag{12}$$

$$\mathrm{DEC} \Longleftrightarrow \rho_{\mathrm{eff}} \geq 0 \text{ and } \rho_{\mathrm{eff}} \pm p_{\mathrm{eff}} \geq 0 \tag{13}$$

Parameters

- In standard mechanics terminology the first four time derivatives of position are referred to as **velocity**, **acceleration**, **jerk** and **snap**.
- In a cosmological setting, in addition to the Hubble parameter $H = \dot{a}/a$, it is appropriate to define the deceleration, jerk, and snap parameters as

$$q = -\frac{1}{H^2}\frac{\ddot{a}}{a}, \qquad j = \frac{1}{H^3}\frac{\dddot{a}}{a}, \quad \text{and} \quad s = \frac{1}{H^4}\frac{\ddddot{a}}{a}. \tag{14}$$

- In terms of the above parameters, we consider the following definitions

$$\dot{H} = -H^2(1+q), \tag{15}$$
$$\ddot{H} = H^3(j+3q+2), \tag{16}$$
$$\dddot{H} = H^4(s-2j-5q-3), \tag{17}$$

respectively.

Weak energy condition

Using these definitions, the weak energy condition takes the following form:

$$\rho_{\rm eff} + p_{\rm eff} = \rho + p + \tfrac{96}{k^2}\Big\{ -(6q^3+27q^2+21q+8qj+9j-s)f''(G) + 24[4(q^2+2q+1)H^2+2q^2+7q+j+4]f'''(G)\Big\}H^8 \geq 0.$$

$$\rho_{\rm eff} = \rho + \frac{1}{2k^2}\Big[-f(G) - 24H^4 q f'(G) - (24)^2 H^8 (2q^2+3q+j) f''(G)\Big] \geq 0.$$

Specific $f(G)$ models

1. Here, we consider realistic models of $f(G)$-gravity, which have been found to reproduce the current acceleration:

$$f_1(G) = \frac{a_1 G^n + b_1}{a_2 G^n + b_2}\,, \qquad (18)$$

$$f_2(G) = a_3 G^n (1 + b_3 G^m)\,, \qquad (19)$$

where a_1, a_2, b_1, b_2, a_3, b_3, n, N and m are constants. In the following, we always assume $n > 0$.

2. **Cautionary note: Work still in progress!**

Example 1: $f_1(G) = \frac{a_1 G^n + b_1}{a_2 G^n + b_2}$

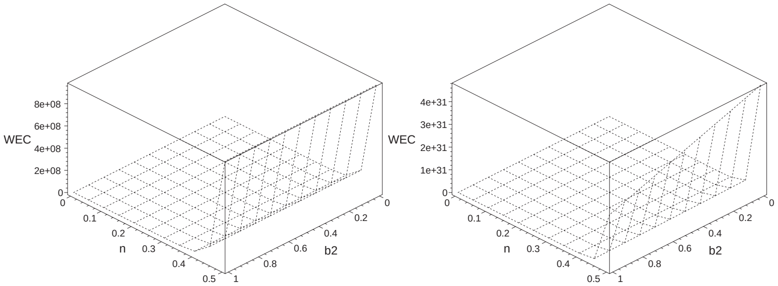

Figure: Weak energy condition: Left plot corresponds to $\rho_{\rm eff} \geq 0$; right plot corresponds to $\rho_{\rm eff} + p_{\rm eff} \geq 0$. We have considered the values $a_1 = -1$, $b_1 = -1$, and $a_2 = 2$.

Example 2: $f_2(G) = a_3 G^n (1 + b_3 G^m)$

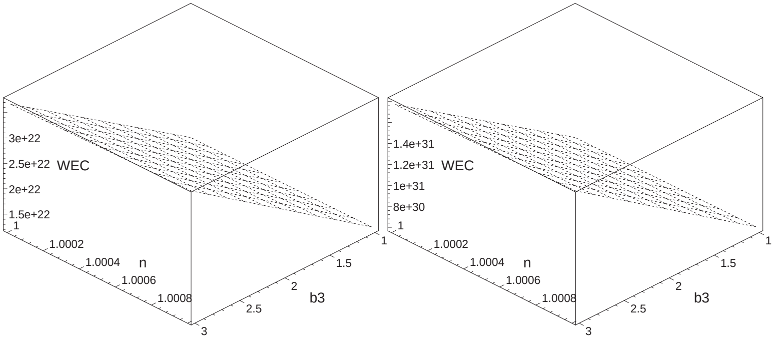

Figure: Weak energy condition: Left plot corresponds to $\rho_{\rm eff} \geq 0$; right plot corresponds to $\rho_{\rm eff} + p_{\rm eff} \geq 0$. We have considered the values $m = 0.5$, and $a_3 = -3$.

Some Conclusions

1. Observations imply acceleration.
2. Theory cannot satisfactorily explain it.
3. GR with dynamical dark energy: no natural model.
4. Modifications to GR – dark gravity: theory gives no natural model.
5. Essential to find viable modified theories of gravity: That explain the Solar System tests and the cosmological epochs.
6. Find additional constraints: For instance, through the Energy Conditions, as outlined in this talk.
7. Theorists need to keep exploring:
   * better models
   * better observational tests