Linear perturbations of a Schwarzschild black hole by thin disc

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- Axially symmetric metric

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu}\mathrm{d}t^2 + r^2 B^2 \mathrm{e}^{-2\nu}\sin^2\theta(\mathrm{d}\varphi - \omega\mathrm{d}t)^2 + \mathrm{e}^{2\zeta-2\nu}\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2\right).$$

- The Energy-momentum tensor of the thin disc

$$T^{\mu}_{\nu} \equiv \mathrm{e}^{2\nu-2\zeta} S^{\mu}_{\nu}(r)\frac{\delta(\cos\theta)}{r},$$

where the surface energy-momentum tensor can be written as

$$S^{\mu}_{\nu} = \sigma u^{\mu} u_{\nu}.$$

- Velocity of the fluid:

$$u^{\mu} = \frac{\mathrm{e}^{-\nu}}{1 - v^2}\left(1, 0, 0, \Omega\right),$$

where $v = r\sin\theta B\mathrm{e}^{-2\nu}(\Omega - \omega)$.

There are only 5 independent Einstein field equations:

- $$\nabla \cdot (r \sin(theta) \nabla B) = 0 \ ,$$
  but B only changes coordinates so it can be chosen.
- $$\nabla \cdot (B \nabla \nu) - \frac{r^2 \sin^2 \theta B^3}{2e^{4\nu}} (\nabla \omega)^2 = 4\pi B \, \frac{\sigma(1+v^2)}{1-v^2} \, \frac{\delta(\cos\theta)}{r} \ ,$$
- $$\nabla \cdot (r^2 \sin^2 \theta B^3 e^{-4\nu} \nabla \omega) = -16\pi r \sin\theta B^2 \frac{\sigma \, v}{e^{2\nu}(1-v^2)} \, \frac{\delta(\cos\theta)}{r} \ ,$$
- The other two contain only first derivatives of ζ and can be used to calculate this metric function. It can be also shown the integrability condition is satisfied for the vacuum space-times.

As a background we will use the Schwarzschild black hole in the isotropic coordinates

$$\mathrm{d}s^2 = -\left(\frac{2r-M}{2r+M}\right)^2 \mathrm{d}t^2 + \left(1+\frac{M}{2r}\right)^4 \left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right),$$

corresponds to

$$\begin{aligned} B &= 1-\frac{M^2}{4r^2}, \\ \nu_0 &= \ln\frac{2r-M}{2r+M}, \\ \omega_0 &= 0, \\ \zeta_0 &= \ln B. \end{aligned} \tag{1}$$

Linearized version of the important equations

$$\nabla\cdot(B\nabla\nu_1) - \frac{r^2\sin^2\theta B^3}{2e^{4\nu_0}}[\nabla\omega_0\cdot\nabla\omega_1 - 4\nu_1(\nabla\omega_0)^2] = 4\pi B\,\frac{\sigma(1+v^2)}{1-v^2}\,\frac{\delta(\cos\theta)}{r}\,,$$

$$\nabla\cdot[r^2\sin^2\theta B^3 e^{-4\nu_0}(\nabla\omega_1 - 4\nu_1\nabla\omega_0)] = -16\pi r\sin\theta B^2\frac{\sigma\, v}{e^{2\nu_0}(1-v^2)}\,\frac{\delta(\cos\theta)}{r}\,,$$

however in the case of static background $\omega_0 = 0$ thus the equations for ν_1 and ω_1 are decoupled.

We will use expansion of the first order perturbation

$$\nu_1 = \sum_{j=0}^{\infty} \nu_{1j}(x) P_j(\cos(\theta)),$$

$$\omega_1 = \sum_{j=0}^{\infty} \omega_{1j}(x)\, T_j^{3/2}(\cos(\theta)),$$

where x is new radial coordinate

$$x \equiv \frac{r}{M}\left(1 + \frac{r^2}{4M^2}\right)$$

The coefficients of the expansion must satisfy

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}\nu_{1l}\right]-l(l+1)\nu_{1l} &= R_l\ ,\\ \frac{\mathrm{d}}{\mathrm{d}x}\left[(x+1)^4\frac{\mathrm{d}}{\mathrm{d}x}\omega_{1l}\right]-\frac{(x+1)^3}{x-1}l(l+3)\omega_{1l} &= S_l\ ,\end{aligned}$$

where S_l and R_l stands for

$$R_l = 2(2l+1)\pi P_l(0) r\sigma(r)\frac{1+v^2}{1-v^2}\ ,$$

$$S_l = -\frac{\pi T_l^{3/2}(0)(2l+3)}{8M^2(l+1)(l+2)}\frac{(2r+M)^3}{(2r-M)}\frac{\sigma(r)v}{1-v^2}\ .$$

The Green's functions of the problem (considering the asymptotic flatness and finity of the perturbation on the horizon) can be found quite easily

$$G_j^{\nu}(x,x') = \begin{cases} -Q_j(x)P_j(x') & \text{for} \quad x > x' \\ -P_j(x)Q_j(x') & \text{for} \quad x < x' \end{cases},$$

where P_j and Q_j stands for the Legendre functions of the first and second kind. Analogously we obtain

$$G_j^{\omega}(x,x') = \begin{cases} -G_j(x)F_j(x') & \text{for} \quad x > x' \\ -F_j(x)G_j(x') & \text{for} \quad x < x' \end{cases},$$

where

$$F_l(x) = {}_2F_1\left(-l, l+3; 4; \frac{x+1}{2}\right)$$

and

$$G_l(x) = F_l(x) \int\limits_x^{\infty} \frac{\mathrm{d}\xi}{[F_l(\xi)]^2(\xi+1)^4}.$$

The Greens functions ale generally enough to formally calculate the perturbation for an arbitrary mass distribution on the disc.
However the real integration is quite complicated and has to be done numerically.
Now we are going to look more closely on the case of the disc consisting of the circulary orbiting particles. In the first order of perturbation it's velocity will be

$$v = \frac{\sqrt{Mr}}{r - \frac{M}{2}}.$$

Inserting this into the r.h.s. of the Einstein's equations we will obtain

$$R_l = 2\pi(2l+1)P_l(0)x\chi(x)\ ,$$

$$S_l = \frac{\pi T_l^{3/2}(0)(2l+3)}{2M(l+1)(l+2)}(x+1)^{\frac{3}{2}}\chi(x)\ ,$$

where $\chi(x) = \frac{r}{x-2}\sigma(r)$. Clearly r.h.s is zero for odd l.

Generally even this case is too complicated to be integrated explicitly. However if $\xi(x)$ is even polynomial it can be done. The particular solutions can be found in the form

$$\nu_{1l}^{Part} = \sum_{j=0}^{\infty} \frac{c_{lj}}{l(l+1) - j(j+1)} P_j(x),$$

where c_{lj} are the coefficients of expansion of R_l into the Legendre polynomials and

$$\omega_{1l}^{Part} = A_l(x)(x+1)^{-\frac{1}{2}} + C_l F_l(x) \ln\left(\frac{\sqrt{x+1}+\sqrt{2}}{\sqrt{x+1}-\sqrt{2}}\right),$$

where C_l are some constants and $A_l(x)$ are polynomials.

These particular solutions still does not describe physically reasonable solutions, because:

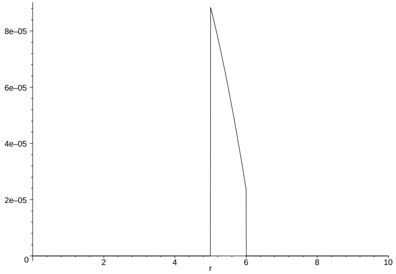
- Describes disc that extends to the black hole horizon, however these is a region, where the stable circular orbit does not exist.
- This solution is not asymptotically flat.

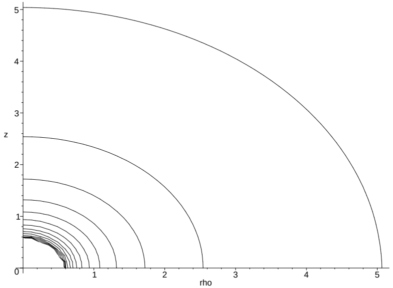
However we can use the "cut and glue" technique restrict this disc to the bound region. More precisely we will consider the disc between radii r_{min} and r_{max} which leads to the perturbation in the form

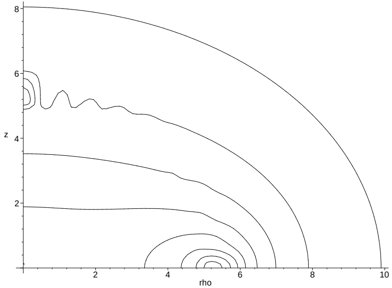
$$\nu_{1l} = \begin{cases} A_{<} P_l(x) & \text{for} \quad x < x_{min} \\ A_D P_l(x) + B_D Q_l(x) + \nu_{1l}^{Part}(x) & \text{for} \quad x_{min} \leq x \leq x_{max} \\ B_{>} Q_l(x) & \text{for} \quad x > x_{max} \end{cases},$$

where $A_{<}$, A_D, B_D and $B_{>}$ are constants determined by requirement of continuity of the function and its first derivations at the rim of the disc. The case of ω_{1l} is analogous.

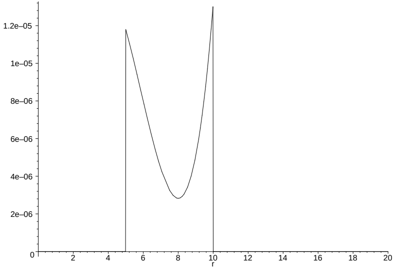
$\chi(x) = \frac{40 - x^2}{100000}$, $M = 1$,
$r_{min} = 5$ and $r_{max} = 6$.
$\rightarrow$: $\sigma(r)$
$\downarrow$: $-\nu(r) = 1.5, 1.4, \ldots, 0.1$
$\searrow$: $\omega(r) \cdot 10^6 = 8, 7, \ldots, 1$

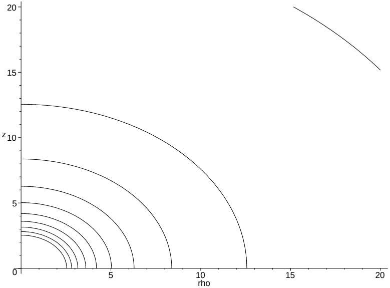


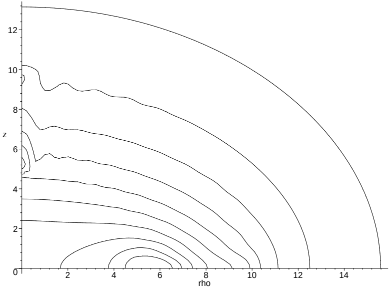




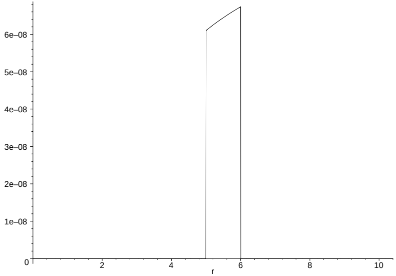
$\chi(x) = \frac{(x^2-60)^2}{100000000} + \frac{100-x^2}{10000000}$,
$M = 1$, $r_{min} = 5$ and
$r_{max} = 10$.
$\rightarrow$: $\sigma(r)$
$\downarrow$: $-\nu(r) = 0.20, 0.18, \ldots, 0.02$
$\searrow$: $\omega(r) \cdot 10^7 = 10, 9, \ldots, 1$

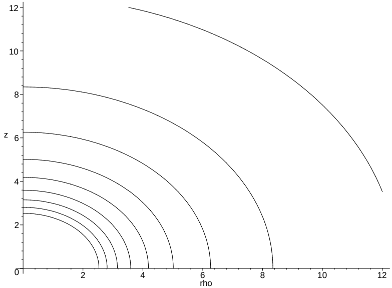


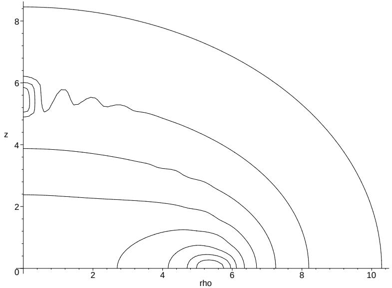




$\chi(x) = \frac{1}{10000000}$, $M = 1$,
$r_{min} = 5$ and $r_{max} = 6$.
$\rightarrow : \sigma(r)$
$\downarrow : -\nu(r) = 0.20, 0.18, \ldots, 0.04$
$\searrow : \omega(r) \cdot 10^9 = 10, 9, \ldots, 1$







Conclusion:

- We were able to obtain first perturbation of the metric functions for a class of mass distributions.
- Because of the nature of harmonic functions it is a problem to obtain higher orders of perturbation in the mass.
- Expansion into the harmonic series simplifies the equations a lot, however it is not suited well for the numerical calculations with the thin disc source.