Vacuum Spacetimes of Embedding Class Two

Alan Barnes

Computer Science
School of Engineering and Applied Science
Aston University, Birmingham UK

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Local Isometric Embedding

- A spacetime is said to be of embedding class N if it can be (locally) isometrically embedded in a flat pseudo-Riemannian manifold of dimension $4 + N$, but in no flat manifold of smaller dimension.

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- Thus a spacetime is of embedding class 2 if it can be embedded in a flat space of signature $(+1, -1, -1, -1, e_1, e_2)$ where $e_i = \pm 1$.
- There are no vacuum spacetimes of embedding class 1 and the only Einstein spaces of class 1 have constant curvature.
- Any spacetime is of embedding class 6 or less.

Class 2 Embedding Equations

The embedding of a spacetime of class 2 is determined by two 2nd fundamental forms Ω_{ab} & Λ_{ab} and a torsion vector t_a satisfying:

- Gauss Equations

$$R^{ab}{}_{cd} = 2e_1\Omega^a_{[c}\Omega^b_{d]} + 2e_2\Lambda^a_{[c}\Lambda^b_{d]}$$

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- Codazzi Equations

$$\Omega^a_{[b;c]} = -e_2\Lambda^a{}_{[b}t_{c]}$$

$$\Lambda^a_{[b;c]} = e_1\Omega^a{}_{[b}t_{c]}$$

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$$\Omega^a_{[b;c]} = -e_2 \Lambda^a{}_{[b} t_{c]}$$
$$\Lambda^a_{[b;c]} = e_1 \Omega^a{}_{[b} t_{c]}$$

- Ricci Equations

$$t_{[a;b]} = -\Omega^c{}_{[a}\Lambda_{b]c}$$

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- The two normals to the embedded spacetime in the enveloping 6-D manifold (and so the 2nd forms) are only determined up to a rotation & reflections if $e_1 e_2 = +1$ or up to a boost & reflections if $e_1 e_2 = -1$.

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- Thus, for $e_1 e_2 = 1$

$$\tilde{\Omega} = \pm(\cos\theta\Omega + \sin\theta\Lambda) \qquad \tilde{\Lambda} = \pm(-\sin\theta\Omega + \cos\theta\Lambda)$$

whereas, for $e_1 e_2 = -1$

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The torsion vector transforms as $\tilde{t}_a = t_a + \theta_{,a}$.

- Thus, if the torsion vector is a gradient, (equivalently if Ω & Λ commute), the torsion vector may be set to zero — 2 Codazzi equations decouple.

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Yakupov's results for Class 2 Vacua

- In 1968 Yakupov proved the useful identity for class 2 vacua:

$$R^{ab}{}_{cd}C^{cd} = 0 \quad \text{where} \quad C_{ab} = \Omega^{e}{}_{[a}\Lambda_{b]e}$$

Thus C is the commutator of Ω & Λ. His proof used the Gauss, Codazzi and Ricci equations plus the Ricci identities.

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- Then, in 1973, he stated without proof two results:
  - For class 2 vacua, the commutator C_{ab} must vanish and hence the torsion vector t^a is a gradient.
  - There are no class 2 vacua of Petrov type III.

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- Then, in 1973, he stated without proof two results:
  - For class 2 vacua, the commutator C_{ab} must vanish and hence the torsion vector t^a is a gradient.
  - There are no class 2 vacua of Petrov type III.
- Most subsequent work has assumed $C_{ab} = 0$. In particular Van den Bergh (1990) confirmed the non-existence class 2 vacua of Petrov type III subject to this assumption.

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Yakupov's Results II

Yakupov gave a few hints, but no details, as to the proof:

- For each Petrov type in turn a canonical basis was chosen for the Riemann tensor.
- Then solutions for the 2 second fundamental forms were obtained compatible with the vacuum conditions $R_{ab} = 0$.
- Use was made of the identity $R^{ab}{}_{cd}C^{cd} = 0$
- For all these solutions the two 2nd fundamental forms commuted.

As far as one can ascertain the proof was purely algebraic following from the Gauss equations, vacuum conditions & the identity $R^{ab}{}_{cd}C^{cd} = 0$.

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Class Two Vacua

- Contracting the Gauss equation the vacuum condition becomes:

$$\Omega^a_c \Omega^c_b - \omega \Omega^a_b + e(\Lambda^a_c \Lambda^c_b - \lambda \Lambda^a_b) = 0$$

where ω and λ are the traces of Ω and Λ respectively and $e = e_1 e_2 = \pm 1$.

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  where ω and λ are the traces of Ω and Λ respectively and $e = e_1 e_2 = \pm 1$.

- Let us call $\Omega^2 - \omega\Omega$ the Ricci square of Ω and similarly the Ricci square of Λ is $\Lambda^2 - \lambda\Lambda$.

- Thus, the Ricci squares of Ω & Λ differ by at most a sign and so *a fortiori* have the same Segré type.

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- Let us call $\Omega^2 - \omega\Omega$ the Ricci square of Ω and similarly the Ricci square of Λ is $\Lambda^2 - \lambda\Lambda$.
- Thus, the Ricci squares of Ω & Λ differ by at most a sign and so *a fortiori* have the same Segré type.
- Moreover, it follows that the Ricci square $\Omega^2 - \omega\Omega$ & Λ commute and so do $\Lambda^2 - \lambda\Lambda$ & Ω.

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Towards a counter-example

- As we will see below, it is easy to find examples where the two second fundamental forms Ω & Λ satisfy the vacuum conditions, but which do not commute.

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- Remarkably all these examples satisfy Yakupov's identity $R^{ab}{}_{cd}C^{cd} = 0$ and its dual $R^{ab}{}_{cd}C^{*cd} = 0$. The identity can be proved using only the Gauss equations and vacuum conditions. It is a purely algebraic constraint.

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- Yakupov's result, if true, cannot follow purely algebraically as he hinted. If these potential counter-examples are to be excluded, it is necessary to consider integrabiity conditions derived from Codazzi and Bianchi identities.

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Some Consequences of Yakupov's Identity

- Yakupov's identity states that C_{ab}, if non-zero, is an eigenbivector of the Riemann tensor with zero eigenvalue. This immediately excludes Petrov types II and D.

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- Yakupov's identity states that C_{ab}, if non-zero, is an eigenbivector of the Riemann tensor with zero eigenvalue. This immediately excludes Petrov types II and D.
- Furthermore Brans (1975) proved that Petrov type I vacuum spaces with a zero eigenvalue do not exist. His proof used the NP formalism and made extensive use of the Bianchi identities and commutator relations.
- Thus we may conclude that the only class 2 vacua with non-zero commutator C_{ab} must be of Petrov type N or III.

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Construction of non-commuting Ω & Λ

- If the Ricci square of Ω (and hence that of Λ) is non-degenerate, then so are Ω & Λ and they both have the same invariant subspace structure and so commute.

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- If the Ricci square of Ω (and hence that of Λ) is non-degenerate, then so are Ω & Λ and they both have the same invariant subspace structure and so commute.
- Hence we need to consider cases where the Ricci squares are degenerate, but where Ω & Λ are not (& cases where they are less degenerate than their Ricci squares).
- Furthermore Ω & Λ must have different invariant subspace structures if they are not to commute.

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Extra Degeneracies of the Ricci Square I

- If Ω is of Segré type [2, 1, 1] with eigenvalues λ, μ & ν resp. its Ricci square is of Segré type
  - $[(1,1),1,1]$ if $\mu+\nu=0$
  - $[2,(1,1)]$ if $\lambda=0$
  - $[(1,1),(1,1)]$ if $\lambda=\mu+\nu=0$

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- If Ω is of Segré type [3, 1] with eigenvalues λ & μ resp. its Ricci square is of Segré type
  - $[(2,1),1]$ if $\lambda+\mu=0$
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- If Ω is of Segré type [1, 1, 1, 1] with eigenvalues λ, μ, ν & σ resp. its Ricci square is of Segré type
  - $[(1,1),1,1]$ if $\nu + \sigma = 0$
  - $[1,1,(1,1)]$ if $\lambda + \mu = 0$
  - $[(1,1),(1,1)]$ if $\lambda + \mu = \nu + \sigma = 0$

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Extra Degeneracies of the Ricci Square II

- If Ω is of Segré type $[Z, \bar{Z}, 1, 1]$ with eigenvalues λ, $\bar{\lambda}$, μ & ν resp. its Ricci square is of Segré type
  - $[(1, 1), 1, 1]$ if $\mu + \nu = 0$
  - $[Z, \bar{Z}, (1, 1)]$ if $\lambda + \bar{\lambda} = 0$
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  - $[Z, \bar{Z}, (1,1)]$ if $\lambda + \bar{\lambda} = 0$
  - $[(1,1),(1,1)]$ if $\lambda + \bar{\lambda} = \mu + \nu = 0$
- There are 17 possible non-commuting combinations compatible with the vacuum conditions. These are all tabulated later.
- However, first the general method of constructing the combinations will be illustrated with two examples.

Potential Counterexample 1

- Ω & Λ are Segré type [211] with Ricci squares [2(11)]

$$\Omega_{ab} = \beta_1 \ell_a \ell_b - \lambda_1 x_a x_b - \mu_1 y_a y_b$$
$$\Lambda_{ab} = \beta_2 \ell_a \ell_b - \lambda_2 \tilde{x}_a \tilde{x}_b - \mu_2 \tilde{y}_a \tilde{y}_b$$

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- (ℓ^a, n^a, x^a, y^a) is a half-null tetrad. $(\tilde{x}^a, \tilde{y}^a)$ is an orthonormal dyad dependent on (x^a, y^a):

$$\tilde{x}^a = \cos\theta x^a + \sin\theta y^a \qquad \tilde{y}^a = -\sin\theta x^a + \cos\theta y^a$$

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- (ℓ^a, n^a, x^a, y^a) is a half-null tetrad. $(\tilde{x}^a, \tilde{y}^a)$ is an orthonormal dyad dependent on (x^a, y^a):
$$\tilde{x}^a = \cos\theta x^a + \sin\theta y^a \qquad \tilde{y}^a = -\sin\theta x^a + \cos\theta y^a$$
- Vacuum conditions:
$$\lambda_1\mu_1 + e\lambda_2\mu_2 = 0, \qquad (\lambda_1 + \mu_1)\beta_1 + e(\lambda_2 + \mu_2)\beta_2 = 0$$
Commutator: $C_{ab} = (\lambda_1 - \mu_1)(\lambda_2 - \mu_2)\sin 2\theta x_{[a} y_{b]}$
Riemann tensor is Petrov type N:
$$\begin{aligned} R^{ab}{}_{cd} = \; & 4\left(\beta_1\lambda_1 + e\beta_2(c^2\lambda_2 + s^2\mu_2)\right) \ell^{[a} x^{b]} \ell_{[c} x_{d]} \\ & -4\left(\beta_1\mu_1 + e\beta_2(s^2\lambda_2 + c^2\mu_2)\right) \ell^{[a} y^{b]} \ell_{[c} y_{d]} \\ & -4ecs\beta_2(\lambda_2 - \mu_2)(\ell^{[a} x^{b]} \ell_{[c} y_{d]} + \ell^{[a} y^{b]} \ell_{[c} x_{d]}) \end{aligned}$$

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Potential Counterexample 2

- Ω & Λ are Segré type [211] with Ricci squares [(11)11]
$$\Omega_{ab} = \lambda_1(\ell_a n_b + n_a \ell_b) + \beta_1 \ell_a \ell_b - \mu_1(x_a x_b - y_a y_b)$$
$$\Lambda_{ab} = \lambda_2(\ell_a n_b + n_a \ell_b) + \beta_2 n_a n_b - \mu_2(x_a x_b - y_a y_b)$$

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Potential Counterexample 2

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  $$\Omega_{ab} = \lambda_1(\ell_a n_b + n_a \ell_b) + \beta_1 \ell_a \ell_b - \mu_1(x_a x_b - y_a y_b)$$
  $$\Lambda_{ab} = \lambda_2(\ell_a n_b + n_a \ell_b) + \beta_2 n_a n_b - \mu_2(x_a x_b - y_a y_b)$$
- Vacuum conditions: $e = -1, \quad \lambda_1 = \lambda_2, \quad \mu_1 = \mu_2$
- Commutator: $C_{ab} = 2\beta_1 \beta_2 \ell_{[a} n_{b]}$
- Riemann tensor is Petrov type I and so excluded by Brans' theorem:

$$R^{ab}{}_{cd} = -4\beta_1 \mu \left(\ell^{[a} x^{b]} \ell_{[c} x_{d]} - \ell^{[a} y^{b]} \ell_{[c} y_{d]}\right) + 4\beta_2 \mu \left(n^{[a} x^{b]} n_{[c} x_{d]} - n^{[a} y^{b]} n_{[c} y_{d]}\right)$$

- $R^{abcd} R_{abcd} = -32\beta_1 \beta_2 \mu^2$

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All Potential Counterexamples

Ω	Λ	Ricci Square	C_{ab}	Petrov Type
[31]	[31]	[(21)1]	$=0$	N
[31]	[211]	[(21)1]	null	O
[211]	[211]	[(11)11] or [(11)(11)]	ST or NS	I
[211]	[211]	[2(11)]	SS	N
[211]	[1111]	[(11)11] or [(11)(11)]	ST or NS	O
[211]	$[Z\bar{Z}11]$	[(11)11] or [(11)(11)]	ST or NS	O
[1111]	$[Z\bar{Z}11]$	[(11)11] or [(11)(11)]	ST	O
$[Z\bar{Z}11]$	$[Z\bar{Z}11]$	[(11)11]	ST	I
$[Z\bar{Z}11]$	$[Z\bar{Z}11]$	$[Z\bar{Z}(11)]$	SS	I
$[Z\bar{Z}11]$	$[Z\bar{Z}11]$	[(11)(11)]	NS	I
[1111]	[1111]	[(11)11]	ST	I
[1111]	[1111]	[11(11)]	SS	I
[1111]	[1111]	[(11)(11)]	NS	I

Type of the bivector C_{ab} : ST = simple timelike, SS = simple spacelike, NS = non-simple

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- In several cases the Riemann tensor vanishes. There (presumably) exist many embeddings with torsion of a flat 4-D spacetime in a flat 6-D flat manifold.
- All, but one, of the remaining cases are Petrov type I and are excluded by Brans' theorem.

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- There are no examples of Petrov type III and so Yakupov's second theorem appears to be valid.
- Yakupov's identity is shown to be a purely algebraic constraint derivable from the Gauss equation and vacuum conditions rather than an integrability condition derived from Codazzi equations etc..

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A Simple Proof of Yakupov's Identity

- As the first step in his proof, Yakupov derived the following equations from the Codazzi equations using the Ricci identities:

$$\Omega^e_{[b} R_{cd]ea} = \Omega_{a[b;cd]} = 2e_2 t_{[c;d} \Lambda_{b]a}$$

$$\Lambda^e_{[b} R_{cd]ea} = \Lambda_{a[b;cd]} = -2e_1 t_{[c;d} \Omega_{b]a}$$

- However, we can derive these equations (with $t_{[a;b]}$ replaced by the commutator C_{ab}) by substituting for R_{abcd} using the Gauss equation in the expressions $\Omega^e_{[b} R_{cd]ea}$ & $\Lambda^e_{[b} R_{cd]ea}$.
- The rest of the proof follows identical lines to Yakupov's (with $t_{[a;b]}$ replaced throughout by the commutator C_{ab})

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