The Einstein-Klein-Gordon-Friedrich system and the non-linear stability of scalar field Cosmologies

Artur Alho, Filipe C. Mena and Juan A. Valiente Kroon

CMAT - Mathematical Physics Group

Queen Mary - Relativity Group

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Asymptotic stability of spatially homogeneous and isotropic spacetimes

The Approach

- Question: do small perturbations of a given solution asymptotically decay to the background solution?
- Within this approach: analyse decay of non-linear perturbations from the knowledge of linear perturbations.

Hyperbolic reduction of EFEs: FOSH systems

- Hyperbolic reductions: The Cauchy problem for the EFEs can be reduced to questions about the Cauchy problem for hyperbolic equations —gauge fixing
- EFEs in vacuum as a *first-order quasi-linear symmetric hyperbolic system* –(H. Friedrich CQG96)

$$\mathbf{A}^0\partial_t\mathbf{u} - \mathbf{A}^j(\mathbf{u})D_j\mathbf{u} = \mathbf{B}(\mathbf{u})\mathbf{u},$$

- Generalization to the Einstein-Euler system using Lagrangian description of the fluid flow and Fermi transport –(H. Friedrich PRD98)

Cosmological context

- Exponential decay of small non-linear perturbations of FLRW backgrounds –(O. Reula PRD99)
- Einstein-scalar field system: Exponential decay of non-linear perturbations of Klein-Gordon-Robertson-Walker backgrounds (KGRW).

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Scalar field and Klein-Gordon equation

- Scalar field $\phi \in \mathcal{C}^{\infty}(\mathcal{M})$. Energy-momentum tensor:

$$\mathbf{T} = \underline{\boldsymbol{\psi}} \otimes \underline{\boldsymbol{\psi}} - \left[\frac{1}{2}\|\boldsymbol{\psi}\|^2_{\mathbf{g}} + \mathcal{V}(\phi)\right]\mathbf{g}$$

$$\underline{\boldsymbol{\psi}} := (\boldsymbol{\nabla}\phi)$$

- Fix the vector-field

$$\boldsymbol{\psi} \equiv \alpha \mathbf{e}_0 \Rightarrow \psi^a = \alpha \delta^a_0$$

and $\|\boldsymbol{\psi}\|^2 = -\alpha^2 \Rightarrow \alpha = \pm\sqrt{-\|\boldsymbol{\psi}\|^2}$.

- For a future-oriented $\boldsymbol{\psi}$, we must choose α to be positive. In a coordinate basis:

$$\psi^{\mu} = \alpha {e_0}^{\mu}, \quad {e_0}^{\mu} = \frac{\nabla^{\mu}\phi}{\sqrt{-\|\boldsymbol{\psi}\|^2}}.$$

- The divergence of the stress-energy tensor gives the evolution equation:

$$N^b\left(\nabla^a T_{ab}\right) = 0 \Rightarrow \mathcal{L}_{\mathbf{e_0}}\alpha + \chi\alpha = \frac{d\mathcal{V}}{d\phi},$$

Lagrangian description and Fermi transport

- The Lagrangian description: the timelike vector of the orthonormal frame follow the matter flow lines. Introduce coordinates $(t, \mathbf{x})$ such that

$$\mathbf{e}_0 = \boldsymbol{\partial}_t, \quad {e_0}^{\mu} = \delta^{\mu}_t.$$

- With this choice: the lapse function is fixed to unity and $D^{\mu}\phi = 0$
- The timelike coframe in terms of the natural cobasis: $\boldsymbol{\theta}^0 = \mathrm{d}t + \beta_j \mathrm{d}x^j$
- Since we are using the scalar field to foliate the spacetime

$$D_{\mu}\phi = 0 \Rightarrow \beta_j = 0 \Rightarrow \chi_{[ab]} = 0.$$

- With this choice

$$\partial_t \phi(t) = \psi(\mathbf{x}, t) := N^a \psi_a = -\alpha < 0, \quad \text{for} \quad \phi > 0 \tag{1}$$

$$\partial_t \psi = -\psi\chi - \frac{d\mathcal{V}}{d\phi}. \tag{2}$$

- The remaining frame components are choosen to be Fermi propagated along $\mathbf{e}_0$

$${\gamma^a}_{b0} = 0$$

Stability approach

- Consider a sequence of smooth initial data $\mathbf{u}^{\varepsilon}$ for the EFEs satisfying the constraints equations on a Cauchy hypersurface Σ.
- Depend continously on the parameter ε, such that, $\varepsilon \to 0$ renders the reference solution $\mathring{\mathbf{u}}$.
- Write the full solution to the EFEs as the Ansatz

$$\mathbf{u}^{\varepsilon} = \mathring{\mathbf{u}} + \varepsilon\breve{\mathbf{u}}^{\varepsilon},$$

  $\breve{\mathbf{u}}^{\varepsilon}$ is a (non-linear) perturbation whose size is controlled by the parameter ε.
- Also

$$\mathbf{B}(\mathring{\mathbf{u}} + \varepsilon\breve{\mathbf{u}}^{\varepsilon}) = \mathring{\mathbf{B}}(\mathring{\mathbf{u}}) + \varepsilon\breve{\mathbf{B}}(\breve{\mathbf{u}}, \varepsilon)$$

- Initial value problem for the non-linear perturbations:

$$\partial_t\breve{\mathbf{u}} = \left[\mathring{\mathbf{A}}^j + \varepsilon\breve{\mathbf{A}}^j(\breve{\mathbf{u}}, \varepsilon)\right] D_j\breve{\mathbf{u}} + \left[\mathring{\mathbf{B}}(t) + \varepsilon\breve{\mathbf{B}}(\breve{\mathbf{u}}, \varepsilon)\right]\breve{\mathbf{u}},$$
$$\breve{\mathbf{u}}(\mathbf{x}, 0) = \breve{\mathbf{u}}_0(\mathbf{x}),$$

Definitions and Stability Theorem

- Stability results for the case where the coefficient of the non-principal part for the linearized system ($\varepsilon = 0$) is a constant matrix are well-known –(Kreiss & Lorenz Acta Numerica7 98).
- These follow from the existence of a negative real part of the eigenvalues for the non-principal part of the linearized system ($\varepsilon = 0$).
- These methods are easily generalized to systems of the type considered here where $\mathring{\mathbf{B}}$ is not constant but depends smoothly on time –(O. Reula PRD99).
- A procedure to analyse stability in the case of systems with vanishing eigenvalues has been given in –(Kreiss, Ortiz & Reula J. Diff. Eq. 98):

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}^{(0)}(t) + \mathbf{u}^{(\lambda)}(\mathbf{x},t), \quad \langle \mathbf{u}^{(0)}, \mathbf{u}^{(\lambda)} \rangle_{L^2(\mathbb{T}^n)} = 0$$

one can write the system as

$$\partial_t \mathbf{u}^{(0)} = \varepsilon \hat{\mathcal{Q}} \left(\breve{\mathbf{A}}^j D_j + \breve{\mathbf{B}}' \right) \mathbf{u}^{(\lambda)} \tag{3}$$

$$\partial_t \mathbf{u}^{(\lambda)} = \left(\mathring{\mathbf{A}}^j D_j + \mathring{\mathbf{B}}' \right) \mathbf{u}^{(\lambda)} + \varepsilon \left(\hat{\mathcal{I}} - \hat{\mathcal{Q}} \right) \left(\breve{\mathbf{A}}^j D_j + \breve{\mathbf{B}}' \right) \mathbf{u}^{(\lambda)} \tag{4}$$

with initial data $\mathbf{u}^{(0)}(0) = \mathbf{u}_0^{(0)}$ and $\mathbf{u}^{(\lambda)}(0,\mathbf{x}) = \mathbf{u}_0^{(\lambda)}(\mathbf{x})$

Klein-Gordon-Robertson-Walker spacetime background

- Metric

$$ds^2 = -dt^2 + \frac{a^2(t)}{1 + \frac{k}{4}r^2} \sum_{j=1}^{3} dx_j^2,$$

- Metric is conformally flat:

$$E_{bd} = B_{bd} = 0.$$

- Foliates the spacetime with the surfaces of constant t

$$\left(\chi^{ST}\right)_{bd} = \left(\chi^{A}\right)_{bd} = 0, \quad a_c = 0.$$

- $\chi(t) = 3\dot{a}/a$
- Friedmann constraint equation for a scalar field

$$\chi^2(t) = -\frac{k}{a^2} + \frac{2}{3}\psi^2(t) - 2\mathcal{V}(\phi),$$

- gauge conditions for the frame are satisfied if

$$\mathbf{e}_0{}^{\mu} = \delta_t^{\mu}, \quad \mathbf{e}_a{}^{\mu} = \left(1 + \frac{k}{4}r\right) a^{-1}\delta_j^{\mu}.$$

,

Linearized system

- Compute $\left.\frac{d\breve{\mathbf{u}}^\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}$ and drop all (non-linear) terms of coupled perturbations.
- Linearized system

$$\begin{aligned}
\partial_t\breve{\phi} &= \breve{\psi} = -|\breve{\psi}| \\
\partial_t\breve{\psi} &= -\mathring{\psi}\breve{\chi} - \mathring{\chi}\breve{\psi} - \left(\frac{d^2\mathring{\mathcal{V}}}{d\phi^2}\right)\breve{\phi} \\
2\partial_t\breve{\chi}_{(bd)} - 2D_{(d}\breve{a}_{b)} &= 2\breve{E}_{bd} - \frac{4}{3}\mathring{\psi}h_{bd}\breve{\psi} - \frac{4}{3}\mathring{\chi}\breve{\chi}_{(bd)} \\
\partial_t\breve{a}_c - D^p\breve{\chi}_{(cp)} &= \left(\frac{2}{3}\mathring{\chi} + \frac{2}{\mathring{\psi}}\frac{d\mathring{\mathcal{V}}}{d\phi}\right)\breve{a}_c \\
\partial_t\breve{E}_{bd} - D_a\breve{B}_{p(d}\varepsilon_{b)}{}^{pa} &= -\frac{1}{3}\mathring{\chi}\breve{E}_{bd} - \frac{1}{2}\mathring{\psi}^2(\breve{\chi}^{ST})_{bd} \\
\partial_t\breve{B}_{bd} - D_a\breve{E}_{p(b}\varepsilon_{d)}{}^{ap} &= -\frac{1}{3}\mathring{\chi}\breve{B}_{bd} \\
\partial_t\breve{e}_b{}^\mu &= \delta_t^\mu\breve{a}_b - \mathring{e}_c{}^\mu\breve{\chi}_b{}^c - \frac{1}{3}\mathring{\chi}\breve{e}_b{}^\mu \\
\partial_t\breve{\gamma}^a{}_{bd} &= -\mathring{\gamma}^a{}_{bp}\breve{\chi}_d{}^p - \frac{1}{3}\mathring{\chi}\breve{\gamma}^a{}_{bd} + \breve{B}_{dp}\varepsilon^{pa}{}_b + \frac{1}{3}\mathring{\chi}\delta_d^a\breve{a}_b - \frac{1}{3}\mathring{\chi}h_{bd}\breve{a}^a
\end{aligned}$$

Analysis of the eigenvalues and stability results

- The characteristic polynomial of $\mathring{\mathbf{B}}(t)$:

$$\lambda\left[\lambda^5 + 5\mathring{\chi}\lambda^4 - \left(4\mathring{\psi}^2 - \frac{28}{3}\mathring{\chi}^2 - \mathring{\mathcal{V}}''\right)\lambda^3 - \left(\frac{32}{3}\mathring{\psi}^2 - \frac{208}{27}\mathring{\chi}^2 - 4\mathring{\mathcal{V}}''\right)\mathring{\chi}\lambda^2 - \left(\frac{64}{9}\mathring{\psi}^2 - \frac{64}{27}\mathring{\chi}^2 - \frac{16}{3}\mathring{\mathcal{V}}''\right)\mathring{\chi}^2\lambda + \frac{64}{27}\mathring{\chi}^3\mathring{\mathcal{V}}''\right] \times \left[\lambda^2 + \frac{5}{3}\mathring{\chi}\lambda + \mathring{\psi}^2 + \frac{4}{9}\mathring{\chi}^2\right]^3 \times \left[\lambda - \frac{2}{3}\frac{\left(3\mathring{\mathcal{V}}' + \mathring{\chi}\mathring{\psi}\right)}{\mathring{\psi}}\right]^3 \left[\lambda + \frac{1}{3}\mathring{\chi}\right]^{23}.$$

- Some roots:

$$\lambda_0 = 0, \quad \lambda_1 = -\frac{1}{3}\mathring{\chi}(t), \quad \lambda_2 = \frac{2}{3}\frac{\left(3\mathcal{V}'(\mathring{\phi}) + \mathring{\chi}\mathring{\psi}\right)}{\mathring{\psi}}.$$

- Negativity of λ_1 and λ_2:

$$\mathring{\chi}(t) > 0 \Rightarrow \text{Ever expanding KGRW spacetime}$$

$$\mathring{\mathcal{V}}'(\mathring{\phi}) > \frac{\mathring{\chi}|\mathring{\psi}|}{3} > 0.$$

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$$\lambda_0=0,\quad \lambda_1=-\frac{1}{3}\mathring{\chi}(t),\quad \lambda_2=\frac{2}{3}\frac{\left(3\mathcal{V}'(\mathring{\phi})+\mathring{\chi}\mathring{\psi}\right)}{\mathring{\psi}}.$$

- Negativity of λ_1 and λ_2:

$$\mathring{\chi}(t)>0 \Rightarrow \text{Ever expanding KGRW spacetime}$$

$$\mathring{\mathcal{V}}'(\mathring{\phi})>\frac{\mathring{\chi}|\mathring{\psi}|}{3}>0.$$

Zero eigenvalue

- $\lambda_0 = 0$: Solutions that tend asymptotically to constant values.
  1. Wave equation into 2 first-order equations: $\breve{\phi} = \phi(t)$, asymptotically converges to the constant value $\breve{\phi}(0) = \breve{\phi}_0$.
  2. ϕ used to foliate the spacetime, and by comparing/similarity with Reula work (no zero eigenvalue).
- Check explicitly:
  1. Find the eigenvector $\breve{\mathbf{u}}^{(0)}$ (general)
  2. check that $\mathring{\mathbf{B}}'(t)\breve{\mathbf{u}}^{(0)} = 0$, with $\left(\breve{\mathbf{u}}^{(0)}\right)^T = [1, 0, \ldots, 0]$
- 38×38 matrix!!

Fifth-order polynomial

- Information about $\mathcal{V}''(\mathring{\phi})$ cannot be deduced from a direct computation of the roots of the remaining fifth-order polynomial.
- Alternatively, we will make use of a result from the so-called *Routh-Hurwitz problem*.

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- A real polynomial with roots whose real part is negative is called a *Hurwitz polynomial* and, for those, the following result holds:

Theorem (Liénard-Chipart)

Let $f(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n$ $(a_0 > 0)$, *be a polynomial with real coefficients. Then the following statements are equivalent:*

(i) *the polynomial is a Hurwitz polynomial;*

(ii) *The coefficients of f are positive and* $\delta_2 > 0, \delta_4 > 0, \ldots, \delta_n, \quad n$ *even;*

(iii) *The coefficients of f are positive and* $\delta_1 > 0, \delta_3 > 0, \ldots, \delta_n, \quad n$ *odd,*

where the Hurwitz determinants are defined by

$$\delta_0 := 1, \quad \delta_l := det \begin{pmatrix} a_1 & a_3 & a_5 & \cdots & a_{2l-1} \\ 1 & a_2 & a_4 & \cdots & a_{2l-2} \\ 0 & a_1 & a_3 & \cdots & a_{2l-3} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_l \end{pmatrix} \quad (l = 1, \ldots, n).$$

- For the second order polynomial the Hurwitz determinants will not give any new condition.
- Now, the positivity of the coefficients of the fifth order polynomial implies

$$\mathring{\mathcal{V}}'' > 0,$$

- while $\delta_2 > 0$ and $\delta_4 > 0$ give

$$\mathring{\mathcal{V}}'' > \frac{28}{3}\left(\mathring{\psi}^2(t) - \frac{263}{63}\mathring{\chi}^2(t)\right)$$

and

$$(\mathring{\mathcal{V}}'')^3 + \frac{4}{3}\left(7\mathring{\chi}^2 - 10\mathring{\psi}^2\right)\left(\mathring{\mathcal{V}}''\right)^2 + \frac{4}{3}\left(31\mathring{\psi}^4 + 4\mathring{\chi}^4 - 80\mathring{\chi}^2\mathring{\psi}^2\right)\mathring{\mathcal{V}}'' + \frac{112}{3}\left(\frac{196}{243}\mathring{\chi}^6 - \frac{280}{81}\mathring{\psi}^2\mathring{\chi}^4 + \frac{31}{7}\mathring{\psi}^4\mathring{\chi}^2 - \mathring{\psi}^6\right) > 0.$$

- The conditions are given in terms of solutions of non-linear evolution ODEs for the background.

Theorem (Main theorem)

Let ϕ be a homogeneous smooth real scalar field with a self-interacting potential $\mathcal{V}(\phi)$ in an expanding Robertson-Walker spacetime and subject to the non-linear Klein-Gordon equation with a potential satisfying the above conditions. Then the KGRW solution is stable in the sense that, given any initial data for small non-linear perturbations $\breve{\mathbf{u}}_0$, whose $\|\breve{\mathbf{u}}\|_{H^k(\mathbb{T}^3)}$ norm is finite, and for $k = 5$ there exists $\varepsilon_0 > 0$, such that for all $0 < \varepsilon \leq \varepsilon_0$, a solution to the non-linear perturbations exists for all times and decays exponentially to zero or converges to constant values.

Some final comments

- With the appropriate modifications in the linearization procedure and inserting a cosmological constant, the symmetric hyperbolic system under study can be used to investigate the non-linear stability of de Sitter spacetime.
- These results can be usefull when considering numerical evolution simulations, since they give conditions for exponential decay and could therefore be used to test the consistency of the numerical steps.

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