Tolman Mass, Generalized Surface Gravity, and Entropy Bounds

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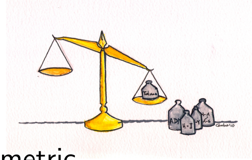
Abstract

- In any static spacetime the quasilocal Tolman mass contained within a volume can be reduced to a Gauss-like surface integral involving the flux of a suitably defined generalized surface gravity.

- By introducing some basic thermodynamics and invoking the Unruh effect, one can then develop elementary bounds on the quasilocal entropy that are very similar in spirit to the holographic bound, and closely related to entanglement entropy.

Tolman mass



In any static spacetime, with the following metric

$$\mathrm{d}s^2 = -e^{-2\Psi}\mathrm{d}t^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{1}$$

the Tolman mass contained in a region Ω is defined [1]

$$m_T(\Omega) = \int_\Omega \sqrt{-g_4}\left(T^0_0 - T^i_i\right)\,\mathrm{d}^3x, \tag{2}$$

where g_4 is the determinant of the $(3+1)$-dimensional metric. Then the Einstein equations imply

$$m_T(\Omega) = \frac{1}{4\pi}\int_\Omega \sqrt{-g_4}\,R^0_0\,\mathrm{d}^3x. \tag{3}$$

Which is a purely geometrical statement.

Now, using the *old* Landau-Liftshitz [2] result,

$$R_0^0 = \frac{1}{\sqrt{-g_4}} \partial_i \left(\sqrt{-g_4}\; g^{0a}\; \Gamma^i_{a0} \right), \tag{4}$$

we can rewrite $m_T(\Omega)$.

Furthermore, consider the following FIDOs, $V^a = \left(\sqrt{|g^{00}|},\, 0,\, 0,\, 0 \right)$, and its acceleration

$$\begin{aligned} A^a &= V^b \nabla_b V^a = V^0 \left(\partial_0 V^a + \Gamma^a_{c0} V^c \right), \\ &= \sqrt{|g^{00}|}\, \Gamma^a_{00} \sqrt{|g^{00}|} = |g^{00}|\, \Gamma^a_{00}. \end{aligned} \tag{5}$$

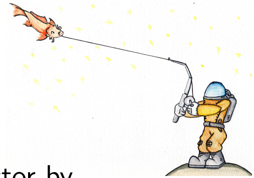
Then

$$
\begin{aligned}
m_T(\Omega) &= \frac{1}{4\pi} \int_\Omega \partial_i \left(\sqrt{-g_4}\, A^i\right) \, \mathrm{d}^3 x, \\
&= \frac{1}{4\pi} \int_\Omega \partial_i \left(\sqrt{g_3}\, e^{-\Psi}\, A^i\right) \, \mathrm{d}^3 x. \qquad (6)
\end{aligned}
$$

And using Stoke's theorem, we have

$$
m_T(\Omega) = \frac{1}{4\pi} \int_{\partial\Omega} \left(e^{-\Psi}\, A^i\right) \, \hat{n}_i \, \sqrt{g_2}\, \mathrm{d}^2 x. \qquad (7)
$$

Generalized surface gravity



Define a (*generalized*) surface gravity 3-vector by

$$\kappa^i = e^{-\Psi} A^i, \tag{8}$$

then its norm,

$$\kappa = \sqrt{g_{ij}\, \kappa^i \kappa^j} = e^{-\Psi} \sqrt{g_{ij}\, A^i A^j}, \tag{9}$$

is just the **usual** surface gravity; the **redshifted** 4-acceleration of the FIDOs.

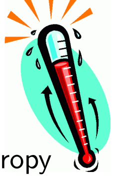
Now, the Tolman mass can be rephrased as

$$m_T = \frac{1}{4\pi} \int_{\partial\Omega} \vec{\kappa} \cdot \hat{n} \, \mathrm{d}\mathscr{A} \leq \frac{1}{4\pi} \, \bar{\kappa}(\partial\Omega) \, \mathscr{A}(\partial\Omega). \tag{10}$$

With

- $\bar{\kappa}(\partial\Omega)$ the average surface gravity,
- $\mathscr{A}(\partial\Omega)$ the total area,
- **no black hole** regions, for now.

Entropy bounds



Consider the Euler (Gibbs-Duhem) relation for the entropy of matter (No black holes, no horizons),

$$s = \frac{\rho + p - \mu\, n}{T}, \tag{11}$$

where we have set $p = \frac{1}{3} tr\{T_{ij}\}$ and $\rho = T_{00}$, as usual. The total entropy is

$$S(\Omega) \;\; = \;\; \int_\Omega \sqrt{g_3}\, \frac{\rho + p + \mu\, n}{T}\, \mathrm{d}^3 x. \tag{12}$$

Now, due to the Tolman [3]

$$T\sqrt{-g_{00}} = T_\infty, \tag{13}$$

and Tolman-Klein [4],

$$\mu\sqrt{-g_{00}} = \mu_\infty, \tag{14}$$

equilibrium conditions, and with $g_{00} \to 1$ at spatial infinity, we have

$$S(\Omega) = \frac{1}{T_\infty}\int_\Omega \sqrt{-g_4}\,(\rho + p)\ \mathrm{d}^3x - \frac{\mu_\infty}{T_\infty}\,N. \tag{15}$$

Since thermodynamical stability requires $\mu \geq 0$, we have

$$\begin{aligned} S(\Omega) &\leq \frac{1}{T_\infty} \int_\Omega \sqrt{-g_4}\,(\rho + p)\,\mathrm{d}^3x, \\ &\leq \frac{1}{T_\infty} \int_\Omega \sqrt{-g_4}\,(\rho + 3p)\,\mathrm{d}^3x, \\ &\leq \frac{m_T(\Omega)}{T_\infty}, \end{aligned} \tag{16}$$

where we have assumed $p \geq 0$. Therefore,

$$S(\Omega) \leq \frac{1}{4\pi\, T_\infty} \bar{\kappa}(\partial\Omega)\,\mathscr{A}(\partial\Omega). \tag{17}$$

Finally, we can invoke the Unruh effect [5] to assert that an observer on $\partial\Omega$ will measure a minimum local Temperature

$$T(x) \geq T_{Unruh}(x) = \frac{||A(x)||}{2\pi}, \tag{18}$$

which when redshifted to infinity implies

$$T_\infty \geq max\left\{\frac{\kappa(x)}{2\pi}\right\}. \tag{19}$$

So the equilibrium temperature of the *object* confined inside $\partial\Omega$ satisfies

$$T_\infty \geq \frac{\bar{\kappa}(\partial\Omega)}{2\pi}. \tag{20}$$

Hence

$$S(\Omega) \leq \frac{\mathscr{A}(\partial\Omega)}{2}. \tag{21}$$

We have used very mild assumptions. This bound relates to

- The holographic bound, $S(\Omega) \leq \frac{1}{4}\mathscr{A}$ [6].
- Bekenstein bound, $S(\Omega) \leq 2\pi\, E(\Omega)\, R(\Omega)$ [7].
- Srednicki's entanglement entropy [8].

Spherical symmetry

As a consistency check, consider a static spherically symmetric geometry,

$$\mathrm{d}s^2 = - e^{-\Phi(r)}\left[1 - \frac{2\,m(r)}{r}\right]\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - 2\,m(r)/r} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \qquad (22)$$

With this particular set of coordinates, assuming asymptotic flatness and normalizing $\Phi(\infty) = 0$,

1. $\sqrt{-g_4} = \sqrt{-g_{00}}\,\sqrt{g_3} = e^{-\Phi}\,r^2 \sin\theta \to 4\pi\,r^2 e^{-\Phi}$.
2. Killing horizon: $2\,m(r_H) = r_H$.
3. Surface gravity, at the horizon: $\kappa_H = \frac{1-2\,m'_H}{2\,r_H}\,e^{-\Phi}$.

Also, the four-acceleration of the FIDOs is

$$A(r) = \frac{m(r) - r\, m'(r)}{r^2 \sqrt{1 - 2\, m(r)/r}} - \Phi'(r) \sqrt{1 - \frac{2\, m(r)}{r}}. \qquad (23)$$

Then, the *generalized* surface gravity takes the form

$$\begin{aligned} \kappa(r) &= \sqrt{-g_{00}} A(r) = e^{-\Phi(r)} \sqrt{1 - 2\, m(r)/r}\, A(r), \\ &= e^{-\Phi(r)} \left[\frac{m(r) - r\, m'(r)}{r^2} - \Phi'(r) \left(1 - \frac{2\, m(r)}{r} \right) \right]. \end{aligned} \qquad (24)$$

This is not the surface gravity of the black hole region, but rather of the *virtual sphere* of radius r. Moreover, a calculation yields [9]

$$m_T(r) = r^2\, \kappa(r). \qquad (25)$$

The entropy inequalities still carry through in essentially the same way,

$$S(r) \leq \frac{m_T(r)}{T_\infty} = \frac{\kappa(r)\, r^2}{T_\infty}. \tag{26}$$

Therefore, considering the FIDOs at radius r, the Unruh effect forces

$$T_\infty \geq \frac{\kappa(r)}{2\,\pi}, \tag{27}$$

so that

$$S(r) \leq 2\,\pi\, r^2. \tag{28}$$

- This bound is *sub-optimal* with respect to the Holographic bound.
- But it is extremely robust and easy to derive.

Discussion

We use very mild assumptions, plus

1. The Einstein equations,
2. The *generalized* surface gravity,
3. The Unruh effect,

to develop a number of bounds on the entropy, that are very minimalist but closely related to

1. The holographic bound [6].
2. generalized 2nd law [7]
3. Bekenstein bound [7].
4. Srednicki's entanglement entropy [8].

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